Supplementary Material for Outlier Rejection for Absolute Pose Estimation with Known Orientation

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1 Solving the Optimization Problem

In this section we show how to solve the following problem,

$$\begin{aligned} \min / \max \quad & \langle \mathbf{x}_0,\, \mathbf{t} \rangle & (1)\\ \text{s.t.} \quad & \langle \mathbf{a}_k,\, \mathbf{t} \rangle \le b_k, \quad k = 1,2,\dots,n_p & (2)\\ & \|\mathbf{X}_i - \mathbf{t}\| \le \alpha \langle \mathbf{x}_i,\, \mathbf{X}_i - \mathbf{t} \rangle . & (3) \end{aligned}$$

From optimization theory it is well known that if the optimal solution exists it must lie at a Karush-Kuhn-Tucker (KKT) point [1]. In this section we explicitly enumerate the KKT points for (1). The Lagrangian is given by

$$\mathcal{L}(\mathbf{t},\mu) = \langle \mathbf{x}_0,\, \mathbf{t} \rangle + \mu_0 \left(\|\mathbf{X_i} - \mathbf{t}\| - \alpha \langle \mathbf{x}_i,\, \mathbf{X_i} - \mathbf{t} \rangle \right) + \sum_{k=1}^{n_p} \mu_k \left(\langle \mathbf{a}_k,\, \mathbf{t} \rangle - b_k \right) \tag{4}$$

and the first order necessary condition is

$$0 = \nabla_{\mathbf{t}} \mathcal{L}(\mathbf{t},\mu) = \mathbf{x}_0 + \mu_0 \left(-\frac{\mathbf{X}_i - \mathbf{t}}{\|\mathbf{X}_i - \mathbf{t}\|} + \alpha \mathbf{x}_i \right) + \sum_{k=1}^{n_p} \mu_k \mathbf{a}_k . \tag{5}$$

First we consider the case when the cone constraint (3) is not active (i.e. $\mu_0 = 0$). Then (5) becomes $\mathbf{x}_0 + \sum_{k=1}^{n_p} \mu_k \mathbf{a}_k = 0$, i.e. $\mathbf{x}_0$ is a linear combination of the plane normals $\mathbf{a}_k$ from the active planar constraints. By the construction of the plane normals it is clear that $\mathbf{x}_0$ can never be formed using only two adjacent plane normals. Thus the active planar constraints are either two non-adjacent planes or more than two planes. In either case the only feasible point is $\mathbf{t} = \mathbf{X}_0$ (i.e. the apex of the approximated cone).

Next assume instead that the constraint in (3) is active (i.e. $\mu_0 \neq 0$). There are four possibilities for the other constraints.

(i) No planar constraints active. ($\mu_k = 0,\ k = 1,2,\dots,n_p$)

(ii) One planar constraint active.

(iii) Two adjacent planar constraints active.

(iv) All planar constraints active.

For the first case (i), equation (5) reduces to

$$\frac{\mathbf{X}_i - \mathbf{t}}{\|\mathbf{X}_i - \mathbf{t}\|} = \mu_0^{-1}\mathbf{x}_0 + \alpha \mathbf{x}_i. \tag{6}$$

Since the left hand side is a unit vector we get the following constraint on μ_0

$$\left\|\mu_0^{-1}\mathbf{x}_0 + \alpha\mathbf{x}_i\right\|^2 = 1 \implies 1 + 2\alpha\mu_0\left\langle \mathbf{x}_0, \mathbf{x}_i\right\rangle + (\alpha^2 - 1)\mu_0^2 = 0, \tag{7}$$

which allows us to determine μ_0. Since $\mu_0 \neq 0$ we can use $\|\mathbf{X}_i - \mathbf{t}\| = \alpha\left\langle \mathbf{x}_i,\ \mathbf{X}_i - \mathbf{t}\right\rangle$ to obtain the translation from (6). Using the constraint it reduces to

$$(I - \alpha(\mu_0^{-1}\mathbf{x}_0 + \alpha\mathbf{x}_i)\mathbf{x}_i^T)(\mathbf{X}_i - \mathbf{t}) = 0. \tag{8}$$

The matrix is invertible since both $(\mu_0^{-1}\mathbf{x}_0 + \alpha\mathbf{x}_i)$ and $\mathbf{x}_i$ are unit vectors and $\alpha < 1$. Thus there is only the (false) solution $\mathbf{t} = \mathbf{X}_i$.

Next we consider the second case (ii) where there is only one active planar constraint. Without loss of generality assume that $\mu_1 \neq 0$ is the active planar constraint. Then (5) reduces to

$$\mathbf{x}_0 + \mu_0\left(-\frac{\mathbf{X}_i - \mathbf{t}}{\|\mathbf{X}_i - \mathbf{t}\|} + \alpha\mathbf{x}_i\right) + \mu_1\mathbf{a}_1 = 0. \tag{9}$$

Since the planar constraint is active the equation $\mathbf{a}_1^T\mathbf{t} = b_1$ is satisfied. By taking the scalar product of (9) with $(\mathbf{a}_1 \times \mathbf{x}_0)$ and using the cone constraint (3) we get

$$(\mathbf{a}_1 \times \mathbf{x}_0)^T\left(I - \alpha^2\mathbf{x}_i\mathbf{x}_i^T\right)(\mathbf{t} - \mathbf{X}_i) = 0, \tag{10}$$

which gives us a second linear constraint on $\mathbf{t}$. From these two linear equations we get a one-parameter family of possible translations $\mathbf{t}$ and intersecting this with the cone in (3) gives us two possible translations.

For the third case (iii) we have two adjacent planes which are both active. This constrains the translation to a line and similarly to the previous case we get two solutions when intersecting with the cone constraint (3).

Finally when all planar constraints are active (iv) the only feasible point is the apex of the (approximated) cone $\mathbf{t} = \mathbf{X}_0$.

So to find the bounds for the projection we simply enumerate all possible KKT points, check feasibility and compute the function values. Since the cone constraint is not differentiable at $\mathbf{t} = \mathbf{X}_i$ we also check this point. Furthermore we must check if the maximization problem is unbounded. For this we simply check if there is an intersection and the the angle between the cones is less than the widths of the cones.

References

[1] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.