Finsler Geodesics Evolution Model for Region based Active Contours

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In this paper, we introduce a new deformable model for image segmentation, by reformulating a region based active contours energy into a geodesic contour energy involving a Finsler metric.

Let $\Omega\subset\mathbb{R}^2$ be the image domain and $\gamma:[0,1]\to\Omega$ be a regular curve with outward normal vector $\mathcal{N}$. Given a function $f:\Omega\to\mathbb{R}$ of interest, we consider the curve evolution scheme:

$$\frac{\partial\gamma}{\partial\tau}=f(\gamma)\mathcal{N}, \tag{1}$$

where τ denotes time. This curve evolution equation can be regarded as a gradient descent, thus a minimization procedure [2], for the functional

$$F(\gamma)=\int_K f(\mathbf{x})\,d\mathbf{x}, \tag{2}$$

where $K\subset\Omega$ is the region inside the closed curve $\gamma:=\partial K$. A complete active contour energy with a curve length regularization can be defined as

$$E(\gamma)=\alpha F(\gamma)+\int_0^1 P(\gamma(t))\|\dot\gamma(t)\|\,dt, \tag{3}$$

where P is an edge based potential function, and $\alpha>0$ is a constant.

Reformulation as Finsler Geodesic Energy: Suppose $\mathcal{V}_\perp:\Omega\to\mathbb{R}^2$ to be a continuously differentiable vector field defined over the domain Ω such that $\mathcal{V}_\perp$ satisfies the following divergence equation:

$$\nabla\cdot\mathcal{V}_\perp(\mathbf{x})=\alpha f(\mathbf{x}),\quad\forall\mathbf{x}\in\Omega, \tag{4}$$

where f is the first order derivative function used in (2) and $\nabla\cdot\mathcal{V}_\perp(\mathbf{x})$ denotes the divergence value of a vector $\mathcal{V}_\perp(\mathbf{x})$. Letting M be the counter-clockwise rotation matrix with rotation angle $\theta=\pi/2$, by divergence theorem, the regional energy F in (2) can be expressed as

$$\begin{aligned}\alpha F(\gamma):=\alpha\int_K f(\mathbf{x})d\mathbf{x}&=\int_K\nabla\cdot\mathcal{V}_\perp(\mathbf{x})\,d\mathbf{x} &(5)\\ &=\int_0^1\langle\mathcal{V}_\perp(\gamma(t)),\mathcal{N}(t)\rangle\,\|\dot\gamma(t)\|\,dt &(6)\\ &=\int_0^1\big\langle M^{\mathrm{T}}\mathcal{V}_\perp(\gamma(t)),M^{\mathrm{T}}\mathcal{N}(t)\|\dot\gamma(t)\|\big\rangle\,dt &(7)\\ &=\int_0^1\big\langle\mathcal{V}(\gamma(t)),\dot\gamma(t)\big\rangle\,dt, &(8)\end{aligned}$$

where $\mathcal{V}=M^{\mathrm{T}}\mathcal{V}_\perp$. Unit vector $\mathcal{N}$ is the outward normal vector of contour γ and $\dot\gamma$ is the tangent vector of γ in **clockwise** order. Indeed, $\mathcal{T}=M^{\mathrm{T}}\mathcal{N}$ is the tangent vector and

$$\dot\gamma(t)=\|\dot\gamma(t)\|\,\mathcal{T}(t)=\|\dot\gamma(t)\|M^{\mathrm{T}}\mathcal{N}(t),\quad\forall t\in[0,1].$$

One can introduce a Finsler metric $\mathcal{F}:\Omega\times\mathbb{R}^2\to\mathbb{R}$:

$$\mathcal{F}(\mathbf{x},\mathbf{u})=P(\mathbf{x})\|\mathbf{u}\|+\langle\mathcal{V}(\mathbf{x}),\mathbf{u}\rangle, \tag{9}$$

which is positive, provided one has the smallness condition [1]:

$$\|\mathcal{V}(\mathbf{x})\|<P(\mathbf{x}),\quad\forall\mathbf{x}\in\Omega. \tag{10}$$

In practice, it is difficult to satisfy the smallness condition (10). Assuming that $\forall\mathbf{x}\in\Omega$, $P(\mathbf{x})\geq 1$, we make use of the following condition:

$$\|\mathcal{V}(\mathbf{x})\|<\min_{\mathbf{y}\in\Omega}\{P(\mathbf{y})\}=1,\quad\forall\mathbf{x}\in\Omega. \tag{11}$$

In view of $\mathcal{F}$ and (5), the energy E (3) is converted to the Finsler geodesic energy:

$$\mathcal{L}(\gamma)=\int_0^1\mathcal{F}(\gamma(t),\dot\gamma(t))dt. \tag{12}$$

Computing the vector field $\mathcal{V}_\perp$ and Finsler Metric $\mathcal{F}$: The minimization procedure of $\mathcal{L}$ (12) is solved inside a neighbourhood U instead of the whole domain Ω. This means that we only require the vector field $\mathcal{V}^\perp$ defined over U. In order to satisfy the smallness condition (10), it is natural to select a solution to (4) minimizing an energy

$$\min\left\{\int_U\|\mathcal{V}_\perp(\mathbf{x})\|^2\,d\mathbf{x}\right\},\quad s.t.\quad\nabla\cdot\mathcal{V}_\perp(\mathbf{x})=\alpha f(\mathbf{x}),\quad\forall\mathbf{x}\in U. \tag{13}$$

Despite the rich regularity for solutions to elliptic PDEs, we could not find a result directly implying that the solution to (13) obeys the desired smallness condition (10). However, such a result can easily be established for a different solution to the divergence equation (4), given by convolution with an explicit kernel

$$\mathcal{V}_\perp(\mathbf{x})=\frac{\alpha}{2\pi}\int_U\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^2}f(\mathbf{y})\,d\mathbf{y}.$$

In that case one indeed obtains using Holder's inequality

$$\begin{aligned}\frac{2\pi}{\alpha}\|\mathcal{V}_\perp(\mathbf{x})\|&\leq\|f\|_q\left(\int_U\frac{1}{\|\mathbf{x}-\mathbf{y}\|^p}d\mathbf{y}\right)^{\frac{1}{p}} &(14)\\ &\leq\|f\|_q\left(\int_{D_U}\frac{1}{\|\mathbf{x}-\mathbf{z}\|^p}d\mathbf{z}\right)^{\frac{1}{p}}=\eta A_U^{\mu}\|f\|_q, &(15)\end{aligned}$$

where D_U is a disk centered around $\mathbf{x}$ and with the same area A_U as U. p,q are two positive constants obeying $1/p+1/q=1$ and $q>2$. η is a constant and $\mu=\frac{1}{p}-\frac{1}{2}>0$. $\|f\|_q$ is the L^q norm of f on U. The condition (11) is thus satisfied when the area of U is sufficiently small.

Finsler Metric Construction: The vector field $\mathcal{V}_\perp$ solution to (13) depends on the neighbourhood U. In order to obtain a vector field obeying $\|\mathcal{V}_\perp\|_\infty<1$, one choose a tubular neighbourhood U with small width hence a small area. On the other hand, U is regarded as the search space for the next evolutional curve. A small U may therefore make the algorithm fall into undesirable local minimas of the geodesic energy $\mathcal{L}$. Thus we use a trick to solve this problem by invoking a non-linear mapping increasing function $T:\mathbb{R}^+\to(0,1)$ defined as $T(x)=1-\exp(-x),\forall x>0$. Thus the new vector field $\bar{\mathcal{V}}$ can be expressed by

$$\bar{\mathcal{V}}(\mathbf{x})=T(\|\mathcal{V}_\perp(\mathbf{x})\|)\,M^{-1}\mathcal{V}_\perp(\mathbf{x})/\|\mathcal{V}_\perp(\mathbf{x})\|,\quad\forall\mathbf{x}\in\Omega. \tag{16}$$

where the smallness condition (11) will be immediately satisfied and M is the counter-clockwise rotation matrix with rotation angle $\theta=\pi/2$. Based on the vector field $\bar{\mathcal{V}}$, the Finsler metric is denoted by $\bar{\mathcal{F}}$ and the geodesic energy $\bar{\mathcal{L}}$ is defined by (12) with $\mathcal{F}:=\bar{\mathcal{F}}$.

The minimization of E (3) is transferred to the minimization of $\bar{\mathcal{L}}$. Note that since in general we induce $\bar{\mathcal{L}}$ with a nonlinear mapping T, there is in fact slight difference in the minimization problems and the results show that our geodesic method is very efficient and robust. The non-linear mapping T is reasonable: **1)** The minimization of E in (5) is relevant to both the directions of γ and the norm of $\mathcal{V}$, i.e., minimizing E is to find a path γ, for which the direction $\gamma'(t)$ for each $t\in[0,1]$ should be as opposite to $\mathcal{V}\big(\gamma(t)\big)$ as possible and the norm $\|\mathcal{V}\big(\gamma(t)\big)\|$ should be as large as possible, giving the relevance between the minimization problems of E and $\bar{\mathcal{L}}$. Introducing T will not modify both goals of the minimization problems. **2)** When the Finsler geodesics evolution scheme tends to stabilize, one can reduce the width of tubular neighbourhood U. Thus $T(\|\mathcal{V}\|)\approx\|\mathcal{V}\|$ as $\|\mathcal{V}\|$ is small.

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