Combination of paths for interactive segmentation - Appendix

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A Self-overlap term

Let $\mathcal{C}$ be a regular curve parameterized over $[0,L]$. Let ϕ be a C^1 function defined over $[0,L]^2$ representing the distance between two positions on the curve:

$$\phi_{\mathcal{C}}(u,v) = \|\mathcal{C}(u) - \mathcal{C}(v)\|^p$$

where p is an arbitrary positive real exponent. The length of the zero level set of $\phi_{\mathcal{C}}$,

$$|\mathcal{Z}_{\mathcal{C}}| = \int_0^L \int_0^L \delta(\phi(u,v)) \, \|\nabla\phi(u,v)\| \, \mathrm{d}u\mathrm{d}v, \tag{1}$$

quantifies the self-overlap of $\mathcal{C}$.

Proposition:
If $\mathcal{C}$ is simple, i.e. without self-intersection and self-tangency, then $|\mathcal{Z}_{\mathcal{C}}| = L\sqrt{2}$

Proof:
As a preliminary calculation, let us express the gradient of ϕ (partial derivatives are written using the indexed notation):

$$\begin{aligned} \nabla\phi(u,v) = & \; [\phi_u(u,v) \;\; \phi_v(u,v)]^T \\ = & \; p\,\|\mathcal{C}(u) - \mathcal{C}(v)\|^{p-2} \begin{bmatrix} \mathcal{C}'(u)\cdot(\mathcal{C}(u) - \mathcal{C}(v)) \\ -\mathcal{C}'(v)\cdot(\mathcal{C}(u) - \mathcal{C}(v)) \end{bmatrix} \end{aligned}$$

If $\mathcal{C}$ is regular and simple, varying with respect to u in range $[0,L]$, $\phi(u,v)$ is nowhere zero except when $u = v$. Hence, for a fixed v, we have:

$$\delta(\phi(u,v)) = \frac{\delta(u-v)}{|\phi_u(v,v)|} \tag{2}$$

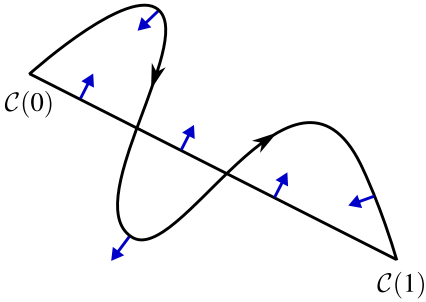

Figure 1: The exteriority of an open curve is measured as the signed area of the multiple connected region that it forms with the line segment joining its two endpoints

Integrating (2) into (1) and applying the definition of measure δ:

$$\begin{aligned}|\mathcal{Z}_{\mathcal{C}}| &= \int_0^L \int_0^L \delta(\phi(u,v))\,\|\nabla\phi(u,v)\|\,\mathrm{d}u\mathrm{d}v \\ &= \int_0^L \int_0^L \frac{\delta(u-v)}{|\phi_u(v,v)|}\,\|\nabla\phi(u,v)\|\,\mathrm{d}u\mathrm{d}v \\ &= \int_0^L \frac{\|\nabla\phi(v,v)\|}{|\phi_u(v,v)|}\mathrm{d}v\end{aligned}$$

Trivially, $\phi(v,v)=0$. However expanding the gradient gives:

$$\begin{aligned}|\mathcal{Z}_{\mathcal{C}}| &= \int_0^L \frac{p\,\|\mathcal{C}(v)-\mathcal{C}(v)\|^{p-2}\sqrt{2(\mathcal{C}'(v)\cdot(\mathcal{C}(v)-\mathcal{C}(v)))^2}}{p\,\|\mathcal{C}(v)-\mathcal{C}(v)\|^{p-2}\,|\mathcal{C}'(v)\cdot(\mathcal{C}(v)-\mathcal{C}(v))|}\mathrm{d}v \\ &= \int_0^L \sqrt{2}\,\mathrm{d}v \\ &= L\sqrt{2}\end{aligned}$$

B Exteriority term

Let $\mathcal{C}$ be a piecewise-smooth regular curve parameterized over $[0,1]$,

$$\mathcal{C} : u \longmapsto \mathcal{C}(u) = [x(u)\ y(u)]^T .$$

If it is simple and positively oriented such that normal vector $\mathcal{C}'^{\perp}$ points inward, its inner area may be expressed using Green's theorem:

$$|\Omega_{\text{in}}(\mathcal{C})| = \frac{1}{2}\int_0^1 \mathcal{C}^{\perp}(u)\cdot\mathcal{C}'(u)\ \mathrm{d}u = \frac{1}{2}\int_0^1 x(u)y'(u)-x'(u)y(u)\ \mathrm{d}u$$

When one calculates the previous expression on a non-simple closed curve, one gets the signed area, in which positively and negatively oriented connected components have positive and negative contributions, respectively.

Proposition:
The signed area formed by an open curve $\mathcal{C}$ over $[0,1]$ and the line segment from $\mathcal{C}(1)$ returning to $\mathcal{C}(0)$ (see Fig. 1), which we use to as the *exteriority* measure in the paper, may

be expressed as:

$$\mathcal{X}[\mathcal{C}] = \frac{1}{2}\int_0^1 \mathcal{C}^\perp \cdot \mathcal{C}' \mathrm{d}u + \frac{1}{2}\mathcal{C}^\perp(1)\cdot\mathcal{C}(0)$$

Proof:

Let S be the parametrization of the line segment joining $\mathcal{C}(1)$ and $\mathcal{C}(0)$, over $[0,1]$:

$$S(u) = (1-u)\mathcal{C}(1) + u\mathcal{C}(0)$$

The signed area is then obtained by applying Green's theorem on a piecewise basis:

$$\begin{aligned}
\mathcal{X}[\mathcal{C}] &= \frac{1}{2}\int_0^1 \mathcal{C}^\perp \cdot \mathcal{C}' \mathrm{d}u + \frac{1}{2}\int_0^1 S^\perp \cdot S' \mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \mathcal{C}^\perp \cdot \mathcal{C}' \mathrm{d}u + \frac{1}{2}\int_0^1 ((1-u)\mathcal{C}^\perp(1) + u\mathcal{C}^\perp(0))\cdot(\mathcal{C}(0)-\mathcal{C}(1))\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \mathcal{C}^\perp \cdot \mathcal{C}' \mathrm{d}u + \frac{1}{2}\int_0^1 (1-u)\mathcal{C}^\perp(1)\cdot\mathcal{C}(0) - u\mathcal{C}^\perp(0)\cdot\mathcal{C}(1)\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \mathcal{C}^\perp \cdot \mathcal{C}' \mathrm{d}u + \frac{1}{2}\int_0^1 (1-u)\mathcal{C}^\perp(1)\cdot\mathcal{C}(0) + u\mathcal{C}^\perp(1)\cdot\mathcal{C}(0)\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \mathcal{C}^\perp \cdot \mathcal{C}' \mathrm{d}u + \frac{1}{2}\int_0^1 \mathcal{C}^\perp(1)\cdot\mathcal{C}(0)\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 \mathcal{C}^\perp \cdot \mathcal{C}' \mathrm{d}u + \frac{1}{2}\mathcal{C}^\perp(1)\cdot\mathcal{C}(0)\mathrm{d}u
\end{aligned}$$