Local Shape Representation in 3D: from Weighted Spherical Harmonics to Spherical Wavelets

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Numerous techniques have been proposed for shape representation, including landmarks [1], medial representation [2], spherical harmonics (SPHARM) [3], weighted SPHARM [4], and spherical wavelets (SW) [5]. Among them, both weighted SPHARM and SW have been used for local shape representation of biological structures. Questions we address in this paper are what is the relationship between them, how to derive SW from weighted SPHARM, how to formulate the derived SW for local shape representation, and which one is better in terms of performance and efficiency for a typical biological problem.

The coordinate x of a point $p(\theta,\varphi)$ on a unit sphere Ω can be represented by weighted SPHARM as the following kernel smoothing

$$x(p)=\int_{\Omega} x(q)K_t^L(p,q)d\eta(q) \qquad (1)$$

where $d\eta(q)=\sin\theta d\theta d\varphi$, $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi)$ are the polar and the azimuthal angles respectively, and the symmetric positive kernel K_t^L is

$$K_t^L=\sum_{l=0}^{L} e^{-l(l+1)t}\sum_{m=-l}^{l} Y_l^m(p)Y_l^m(q)=\sum_{l=0}^{L}\frac{2l+1}{4\pi}e^{-l(l+1)t}P_l(p\cdot q) \qquad (2)$$

where Y_l^m is SPHARM with the degree $l\geq 0$ and order $|m|\leq l$. Eq. (2) is essentially the Gauss-Weierstrass kernel [6].

The term $e^{-l(l+1)t}$ in Eq. (2) can be considered as a discretized version of the continuously defined function $\varphi_0(u)=e^{-tu(u+1)}$. The dilation of φ_0 is given as

$$\varphi_j(u)=D_j\varphi_0(u)=\varphi_0(2^{-j}u) \qquad (3)$$

where D_j is called dilation operator of j-th level. A system of scale discrete scaling function can be generated via φ_0 and its dilations φ_j as

$$\Phi_j(v)=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\varphi_j(l)P_l(v), v\in[-1,1] \qquad (4)$$

The discrete scaling function of Eq. (4) defines a "discrete approximate identity" [7] in $L^2(\Omega)$. Based on Eq. (4), scale discrete wavelets on the sphere can be introduced as the difference of two successive resolution levels

$$\Psi_j(v)=\Phi_{j+1}(v)-\Phi_j(v)=\sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\left(\varphi_{j+1}(l)-\varphi_j(l)\right)P_l(v) \qquad (5)$$

which can be considered as a difference-of-Gaussian (DoG) wavelet.

As discussed in the paper, the SW derived above are poorly localized, and in fact they do not really resemble wavelets. We propose a new way to construct over-complete SW based on the group theoretic approach [8], and use the theoretical results from the work of [9] to build self-invertible filter banks, which are employed for decomposing and reconstructing images.

We construct the spherical DoG wavelet by projecting its Euclidean planar formula on to the sphere

$$DoG(\theta,\varphi)=\frac{1}{2\pi}(1+\tan^2(\theta/2))\left(\frac{1}{\sigma_1^2}e^{-\frac{2}{\sigma_1^2}\tan^2\frac{\theta}{2}}-\frac{1}{\sigma_2^2}e^{-\frac{2}{\sigma_2^2}\tan^2\frac{\theta}{2}}\right) \qquad (6)$$

The term $1+\tan^2(\theta/2)$ is to ensure the unitarity of the projection. The nth analysis filters $\tilde{h}_n$ of the self-invertible filter banks are the stereographic dilation [8] of Eq. (6):

$$\tilde{h}_n(\theta,\varphi)=\left(\prod_{i=1}^{n}b_i\right)D_{a_n}DoG(\theta,\varphi) \qquad (7)$$

where b_i are the amplitude scaling parameters that control the tradeoff between self-invertibility and norm-preserving dilation, and D_{a_n} is the stereographic dilation operator.

A spherical continuous wavelet transform of $x(\theta,\varphi)$ can be given in terms of a wavelet basis by the projection on to each wavelet basis function by spherical convolution

$$W_n(\alpha,\beta,\gamma)=\int_{\Omega}[R(\alpha,\beta,\gamma)\tilde{h}_n]*(\theta,\varphi)x(\theta,\varphi)d\Omega \qquad (8)$$

where $R(\alpha,\beta,\gamma)$ is the rotation operator. To produce reconstructed surface components, the synthesis filters are used to project a function in $L^2(SO(3))$ onto $L^2(\Omega)$ by inverse convolution

$$\hat{x}_n(\theta,\varphi)=\int_{SO(3)}[R(\alpha,\beta,\gamma)h_n](\theta,\varphi)W_n(\alpha,\beta,\gamma)d\rho \qquad (9)$$

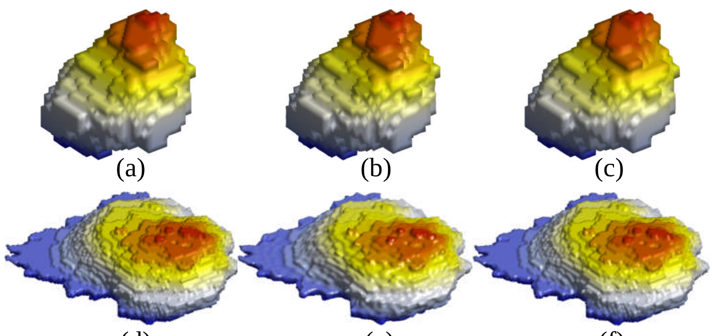

Figure 1: Shape representations of example surfaces of a left amygdala (a) and a neutrophil cell (d) via both weighted SPHARM with 78 degree ((b) and (e)) and SW with level 7 ((c) and (d)), repectively.

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