Fast and Robust Surface Normal Integration by a Discrete Eikonal Equation

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Since the integration of normal vectors plays an important role for reconstructing a surface, over decades it has been one of the most fundamental problems in computer vision and thereby extensively investigated by many researchers [6]. While many schemes have been proposed, there is, however, still a need for methods that combine accuracy, robustness and high efficiency. In view of efficiency, the fast marching (FM) [1, 3] method appears to be a natural candidate for an algorithmic approach, because the method gives us a complexity of $\mathcal{O}(N \log N)$, where N is the number of pixels of the computational domain, for the problems described by a static eikonal-type equation. In the work of Ho et al. [2] this strategy has been adopted, which is based on an analytic formulation of the integration task in terms of an eikonal equation. Whereas in [2] some promising results are presented, the authors also report significant problems with the robustness and accuracy of the scheme.

In this paper, we improve the scheme of Ho et al. [2] by proposing a complete discrete formulation (DEFM) in terms of a proper approximation of the underlying partial differential equation (PDE). Furthermore, by relying on pre-computed geodesic distance as a metric on the computational domain we extend our method in such a way that the DEFM can handle topologically more challenging domains, e.g. domains with holes.

From the fundamental theorem of calculus an antiderivative v in 1D is given by $\int v'(x_1)\,dx_1 = v(x_1) + c$ with a constant c. In 2D, this can be extended as

$$w(x_1,x_2) := v(x_1,x_2) + \lambda f(x_1,x_2)\,, \tag{1}$$

where $\lambda > 0$ is a constant parameter and f denotes a function. Since a function f in (1) should not change the important structure of w, specially critical points, in [2] as such a function

$$f_{\text{Ho}} := x_1^2 + x_2^2 \tag{2}$$

is chosen which admits only one minimum at origin. For the deployment of FM, the expression in (1) is turned into an eikonal-type expression

$$|\nabla w| = |\nabla v + \lambda \nabla f_{\text{Ho}}| = \sqrt{(v_{x_1} + \lambda\, 2x_1)^2 + (v_{x_2} + \lambda\, 2x_2)^2} \tag{3}$$

with $v_{x_1} := \frac{\partial v}{\partial x_1}$ and $v_{x_2} := \frac{\partial v}{\partial x_2}$. Since all elements on the right hand side of (3) are known, the FM method allows to compute w from the PDE $|\nabla w| = |\nabla v + \lambda \nabla f_{\text{Ho}}|$. In the method of Ho et al. [2], the analytic formulation of ∇f_{Ho} in (2) is employed. However, since the analytic formulation has the same effect as the central difference method, the result by this method suffers from severe instability for solving (3) by the FM, see Figure 1(a).

In view of the properties from the underlying eikonal-type PDE and FM method, our main advancement stems from the deployment of a proper discretisation for (3) – *upwind scheme* [5]. In 1D, this upwind discretisation reads as

$$\hat{f}_x \;:=\; \max\left(D^- f, -D^+ f, 0\right) \tag{4}$$

with

$$D^- f = \frac{f_i - f_{i-1}}{\Delta x} > 0 \qquad \text{and} \qquad D^+ f = \frac{f_{i+1} - f_i}{\Delta x} < 0 \tag{5}$$

where Δx is the mesh width and f_j denotes a function value at a grid point $j \in \mathbb{Z}$. Each inequality in (5) holds for consistency since the upwind scheme chooses only one direction for the propagation of the information.

Our scheme analysis based on [4] shows that the proposed method is monotone and thereby stable if

$$\lambda \geq \varepsilon > 0\,, \tag{6}$$

where ε is a very small pre-defined constant. This suggests that the proposed method gives us no restrictions for the choice of the parameter λ in (3) in contrast to the case of Ho et al.

(a) Optimal result by the scheme of Ho et al. (b) Generic result by our method.

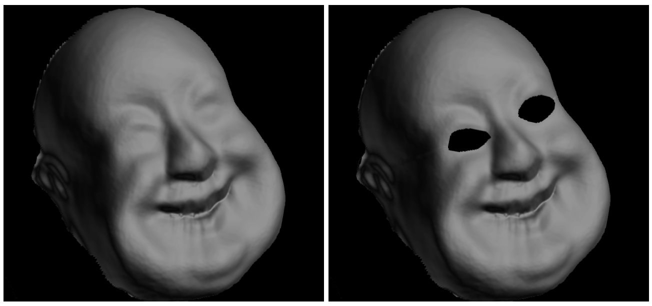
Figure 1: Reconstruction results by each method.

As shown in Figure 1 and Table 1, numerical experiments confirm our analysis in that even with very large λ values the present result outperforms in all error measures.

Table 1: Error measurements for Lena experiment shown in Figure 1.

	Mean	Median	Standard deviation
Ho et al. ($\lambda = 0.2$)	0.3060	0.2079	0.3604
Our method ($\lambda = 1000000$)	0.0785	0.0364	0.1325

Moreover, in order to deal with topologically more challenging computational domains we employ the more general *geodesic distance* for the function f in (1) instead of L_2 metric given in (2). Our numerical experiment again verifies that the geodesic measurements can handle non-trivial integrations domains accordingly as shown in Figure 2.



(a) Reconstruction without a mask. (b) Reconstruction with a mask.

Figure 2: Renderings of the Buddha face.

[1] J.A. Sethian. *Level Set Methods and Fast Marching Methods*. Cambridge University Press, 2nd edition, 1999.

[2] J. Ho and J. Lim and M.-H. Yang and D.J. Kriegman. Integrating surface normal vectors using fast marching method. In *Proc. ECCV*, 239–250, 2006.

[3] J.N. Tsitsiklis. Efficient algorithms for globally optimal trajectories. *IEEE T-Automatic Control*, 40 (9): 1528–1538, 1995.

[4] R.J. LeVeque. *Numerical Methods for Conservation Laws*. Birkhäuser, 1992

[5] E. Rouy and A. Tourin. *A viscosity solutions approach to shape-from-shading*. *SINUM*, 29(3): 867–884, 1992.

[6] J.-D. Durou and J.-F. Aujol and F. Courteille *Integrating the normal field of a surface in the presence of discontinuities*. In *Proc. EMMCVPR*, 261–273, 2009.