Supplementary Document:
Compressive Inverse Light Transport

1 Lemma 3.1 (ℓ_0)

In this section, we shall prove that, in the backward problem, B must satisfy exactly the same condition as that in the forward problem, i.e, the null space property.
Let Σ_k be the set of all k-sparse vectors. We denote A_j to be the jth column of the $n \times N$ measurement matrix A, and $B = AQ$ to be the observation matrix. We also use u_j to denote the jth column of Q^{-1}, formally, $Q^{-1} = [u_1, u_2, ..., u_j, ..., u_N]$. Consider the problem,

$$P_0^{-1}(j) : \ \min \|x\|_0 \ subject\ to\ Bx = A_j \tag{1}$$

We denote the minimizer of this problem as $\hat{x}_j$ implying its dependence on A_j. We will show that, for the minimizer $\hat{x}_j$ of $P_0^{-1}(j)$ to be the jth column of Q^{-1}, a necessary and sufficient condition is for B to satisfy the null space property, i.e. $\ker(B) \cap \Sigma_{2k} = \{0\}$.

Lemma 1.1. *Let $u_j \in \Sigma_k$ be the jth column of Q^{-1}. There exist a unique solution $\hat{x}_j$ to the minimization procedure $P_0^{-1}(j)$ such that $\hat{x}_j = u_j$,*
if and only if

$$\ker(B) \cap \Sigma_{2k} = \{0\}$$

Proof. "if":
We notice the fact that, $u_j \in \Sigma_k$ is a solution to the equation $Bx = A_j$, i.e,

$$Bu_j = BQ^{-1}e_j = AQQ^{-1}e_j = Ae_j = A_j \tag{2}$$

Suppose there exist another solution $\hat{x}_j \in \Sigma_k \backslash \{u_j\}$ satisfies $Bx = A_j$, i.e,

$$B\hat{x}_j = A_j = Bu_j \Leftrightarrow B(\hat{x}_j - u_j) = 0 \tag{3}$$

We denote $\Delta y_j = \hat{x}_j - u_j \in \Sigma_{2k}$. On the other hand, Eqn **??** implies $\Delta y_j \in \ker(B)$, therefore,

$$\Delta y_j \in \ker(B) \cap \Sigma_{2k}$$

Since $\ker(B) \cap \Sigma_{2k} = \{0\}$, we have,

$$\Delta y_j = 0 \Leftrightarrow \hat{x}_j = u_j$$

"only if":
Since $u_j \in \Sigma_k$ is a solution to $Bx = A_j$ as noticed in Eqn **??**, the uniqueness of solution to $P_0^{-1}(j)$ states that,

$$\begin{aligned} \forall \hat{x}_j \in \Sigma_k \backslash u_j,\ B\hat{x}_j \neq Bu_j \\ \forall \hat{x}_j \in \Sigma_k \backslash u_j,\ B(\hat{x}_j - u_j) \neq 0 \end{aligned} \tag{4}$$

Set $\Delta y_j = \hat{x}_j - u_j \in \Sigma_{2k} \backslash \{0\}$, Eqn **??** is equivalent to say,

$$\forall \Delta y_j \in \Sigma_{2k} \backslash \{0\},\ \Delta y_j \in \overline{\ker(B)}$$

This is equivalent to,

$$\ker(B) \cap \Sigma_{2k} = \{0\}$$

□

The null space property is very intuitive, it says that, for all non-$2k$-sparse signals, B has to take them to 0, so that for the non-zero compressions on the right hand side, there must be some sparse vector corresponds to it.

1.1 Theoretical conditions (ℓ_0)

Lemma **??** in section **??** establishes an equivalent condition in observation matrix B. This enables us to concentrate our study on $B = AQ$. We shall soon see its relationships with the RIP as well as $Incoherence$ in later sections. Our primary goal in this section is to develop analytical basics for establishing condition on A and Q. We introduce a necessary and sufficient condition relating the Gram Matrix $B_T^t B_T$ and the null space property in this section and develop the extended null space property under $B = AQ$. *Note that all statements in this section dictates what is theoretically possible and does not concern practical implementations. The null space property and extended null space property are not easy to check if given specific matrices. We do not worry about practicality of l1-implementation at the current stage, so now we have more freedom.*

1.2 Gram matrix $B_T^t B_T$

Lemma 1.2. *Given a set $T \subseteq \{1, 2, ...N\}$, we define the matrix B_T as the one formed from B by using columns from the set T. All eigenvalues of the Gram matrix $B_T^t B_T$ are positive for all $\{T||T| \leq 2k\}$ if and only if* $\ker(B) \cap \Sigma_{2k} = \{0\}$

Proof. For any set T, since $B_T^t B_T$ is square, we are able to define $|T|$ eigen-pairs $\{(\lambda_i, y_i)\}_{i=1:|T|}$ with all eigenvalues of $B_T^t B_T$ non-negative because for any eigen-pair (λ_i, y_i), we have

$$\lambda_i = \frac{y_i^t B_T^t B_T y_i}{y_i^t y_i} = \frac{\|B_T y_i\|_2^2}{\|y_i\|_2^2} \geq 0 \tag{5}$$

On the other hand, for any $x \in \Sigma_{2k}$ with x_T be the vector that has non-zero support T.We have, $x \notin \ker(B)$, i.e. $Bx = B_T x_T \neq 0$. Isolate out the elements on support T ,we have

$$B_T x_T \neq 0, \forall x_T \neq 0$$

Combine with Eqn **??**, we can prove that, $\{\lambda_i\}_{i=1:|T|} > 0$ for all $\{T||T| \leq 2k\}$. □

We can state other equivalent conditions to Lemma **??**: Since $x_T = 0$ is the only solution to the over-determined system:$B_T x_T = 0$, therefore the columns of B_T are linearly independent, which means that the dimension of the column space of B_T is $|T|$. Also, If we write B_T in a SVD form, we can easily observe that B_T and $B_T^t B_T$ have the same number of non-zero singular values which is T. Since the dimension of $B_T^t B_T$ is $T \times T$, we have that $B_T^t B_T$ is square, full rank, and invertible, and again we emphasize that, these are all *necessary and sufficient* conditions for Q^{-1} to be reconstructible. Expand on $B = AQ$, we have $B_T = AQ_T$ and $B_T^t B_T = Q_T^t A^t A Q_T$. If matrix A is approximate orthogonal, $A^t A \sim I$, then the condition requires that $Q_T^t Q_T$ is approximately identity.

1.3 Extended null-space property (ℓ_0)

Proposition 1.1. *If A is $m \times n$ and Q is $n \times p$, then*

$$rank(AQ) = rank(Q) - \dim(\ker(A) \cap Range(Q)). \tag{6}$$

Lemma 1.3. *Suppose a $n \times |T|$ matrix B_T with full column rank is a multiplication of a $n \times N$ matrix A and a $N \times |T|$ matrix Q_T with*

$$2|T| \leq n < N$$

Then

$$\ker(A) \cap Range(Q_T) = \{0\}$$

Proof. Since the columns of B_T to be linearly independent, this means that

$$rank(B_T) = rank(AQ_T) = |T|$$

Since $rank(Q_T) \leq |T| = \min(|T|, N)$ and $\dim(\ker(A) \cap Range(Q_T)) \geq 0$, by Proposition **??**, we must have

$$rank(Q_T) = |T| \quad and \quad \dim(\ker(A) \cap Range(Q_T)) = 0$$

Since in our case Q is invertible and thus $rank(Q_T) = |T|$, for $|T|$-sparse vectors to be reconstructible, the sufficient and necessary condition is

$$\ker(A) \cap Range(Q_T) = \{0\}, \forall T$$

$\square$

This means for all $|T|$-dimensional subspaces spanned by columns of matrix Q, A should not take vectors in those subspaces to zero.

2 Theorem 3.1

2.1 Linear mapping transforms Concentration Inequality

Proposition 2.1. *(Concentration inequality, a.k.a Norm preservation) Let $x \in \mathbb{R}^N$. For $n \times N$ random matrices A whose entries are independent Gaussian random variables,*

$$A_{i,j} \sim \mathcal{N}(0, \frac{1}{n})$$

or Bernouli random variables,

$$A_{i,j} := \begin{cases} +1/\sqrt{n} & \text{with probability 1/2} \\ -1/\sqrt{n} & \text{with probability 1/2} \end{cases}$$

Then

$$Pr(|\|Ax'\|_2^2 - \|x'\|_2| \geq \epsilon \|x'\|_2) \leq 2e^{-nc(\epsilon)}, \quad 0 < \epsilon < 1 \tag{7}$$

where $c(\epsilon) = \epsilon^2/4 - \epsilon^3/6$.

If we let $x' = Qx$, immediately we have

Corollary 2.1. *Let $x \in \mathbb{R}^N$. For $n \times N$ random matrices A whose entries are independent Gaussian random variables,*

$$A_{i,j} \sim \mathcal{N}(0, \frac{1}{n})$$

or Bernouli random variables,

$$A_{i,j} := \begin{cases} +1/\sqrt{n} & \text{with probability 1/2} \\ -1/\sqrt{n} & \text{with probability 1/2} \end{cases}$$

Then

$$Pr(|\|AQx\|_2^2 - \|Qx\|_2| \geq \epsilon \|Qx\|_2) \leq 2e^{-nc(\epsilon)}, \quad 0 < \epsilon < 1 \tag{8}$$

where $c(\epsilon) = \epsilon^2/4 - \epsilon^3/6$.

2.2 Covering Numbers - Bridging RIP and $Concentration\ Inequality$

Lemma 2.1. *(Covering numbers) If $\mathcal{S}$ is the unit sphere of $\mathbb{R}^N$ relative to an arbitrary norm $\|\cdot\|$, then there exists a set $\mathcal{U} \in \mathcal{S}$ with*

$$\forall z \in \mathcal{S}, \quad \min_{u \in \mathcal{U}} \|z - u\| \leq \xi, \quad \textit{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2}{\xi}\right)^N \tag{9}$$

Proof. Let $\{u_1, ..., u_h\} \subseteq \mathcal{S}$ be a set of h points on the sphere $\mathcal{S}$ such that $\|u_i - u_j\| > \xi$ for all $i \neq j$. We are able to choose h as large as possible because as h increases, we can choose smaller ξ. Therefore, we can always find a set of $\mathcal{U}$ to ensure the following is true,

$$\forall z \in \mathcal{S}, \quad \min_{i \in [1:h]} \|z - u_i\| \leq \xi$$

Let $\mathcal{B}$ be the unit ball of $\mathbb{R}^N$ endowed with the norm $\|\cdot\|$, i.e. $\{\mathcal{B} : \forall v, \|v\| \leq 1\}$. Since $\|u_i - u_j\| > \xi, \forall i \neq j$, we must have

$$[u_i + \frac{\xi}{2}\mathcal{B}] \bigcap [u_j + \frac{\xi}{2}\mathcal{B}] = \emptyset, \forall i \neq j$$

The above can also be obtained by a proof-by-contradiction argument. Also we have

$$u_i + \frac{\xi}{2}\mathcal{B} \subseteq \left(1 + \frac{\xi}{2}\right)\mathcal{B}, \forall i$$

because $u_i \in \mathcal{S}$ is on the unit sphere of $\mathbb{R}^N$, for any $z \in \mathcal{B}$, we have

$$\|u_i + \frac{\xi}{2}z - 0\| \leq \|u_i\| + \frac{\xi}{2}\|z\| \leq 1 + \frac{\xi}{2}$$

Now we add-up the volume of all the balls $u_i + \frac{\xi}{2}\mathcal{B}$, by the inclusion argument from above, we get the inequality

$$h\Omega(\frac{\xi}{2}\mathcal{B}) = \sum_{i=1}^{h} \Omega(u_i + \frac{\xi}{2}\mathcal{B}) \leq \Omega\left((1 + \frac{\xi}{2})\mathcal{B}\right)$$

Then by the N-homogeneity of volume

$$h\left(\frac{\xi}{2}\right)^N \Omega(\mathcal{B}) \leq \left(1 + \frac{\xi}{2}\right)^N \Omega\left(\mathcal{B}\right)$$

automatically

$$h \leq \left(1 + \frac{2}{\xi}\right)^N$$

□

Corollary 2.2. *If $\mathcal{S}$ is the unit sphere of $\mathbb{R}^N$ relative to Euclidean norm $\|\cdot\|_2$, then there exists a set $\mathcal{U} \subseteq \mathcal{S}$ with*

$$\forall z \in \mathcal{S}, \quad \min_{u \in \mathcal{U}} \|Qz - Qu\|_2 \leq \xi, \quad \textit{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2\sigma_Q}{\xi}\right)^N \tag{10}$$

Or equivalently,

$$\forall z \in \mathcal{S}, \quad \min_{u \in \mathcal{U}} \|Qz - Qu\|_2 \leq \sigma_Q \xi', \quad \textit{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2}{\xi'}\right)^N \tag{11}$$

Proof. Let $\{u_1, ..., u_h\} \subseteq \mathcal{S}$ be a set of h points on the sphere $\mathcal{S}$ such that $\|u_i - u_j\|_2 > \xi'$ for all $i \neq j$. As stated in Lemma **??**, we can always find a set $\mathcal{U}$ to ensure the following is true,

$$\forall z \in \mathcal{S}, \quad \min_{u \in \mathcal{U}} \|z - u\|_2 \leq \xi', \quad \text{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2}{\xi'}\right)^N \tag{12}$$

which means that we can always find a set $\mathcal{U}$ to ensure that the following is true

$$\forall z \in \mathcal{S}, \quad \min_{u \in \mathcal{U}} \|Qz - Qu\|_2 \leq \sigma_Q \xi', \quad \text{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2}{\xi'}\right)^N \tag{13}$$

this is because the following is always true:

$$\|Q(z-u)\|_2 \leq \sigma_Q \|(z-u)\|_2$$

set $\xi = \sigma_Q \xi'$, we have

$$\forall z \in \mathcal{S}, \quad \min_{u \in \mathcal{U}} \|Qz - Qu\|_2 \leq \xi, \quad \text{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2\sigma_Q}{\xi}\right)^N \tag{14}$$

□

Corollary 2.3. *Let K be a fixed index set of cardinality $|K| = k$. If $\mathcal{S}_{\Sigma_K}$ is the unit sphere of Σ_K relative to Euclidean norm $\|\cdot\|_2$, then there exists a set $\mathcal{U} \subseteq \mathcal{S}_{\Sigma_K}$ with*

$$\forall z \in \mathcal{S}_{\Sigma_K}, \quad \min_{u \in \mathcal{U}} \|Qz - Qu\|_2 \leq \xi, \quad \textit{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2\sigma_Q}{\xi}\right)^k \tag{15}$$

or equivalently

$$\forall z \in \mathcal{S}_{\Sigma_K}, \quad \min_{u \in \mathcal{U}} \|Q_K z - Q_K u\|_2 \leq \xi, \quad \textit{and} \quad |\mathcal{U}| \leq \left(1 + \frac{2\sigma_{Q_K}}{\xi}\right)^k \tag{16}$$

where Q_K is the sub-matrix of Q by picking its columns whose indices are in the set K.

Lemma 2.2. *Let A be a random matrix of size $n \times N$ drawn according to any distribution that satisfies the concentration inequality* **??**. *Then, for any set K with $|K| = k$ and any $0 < \delta < 1$, we have*

$$(1-\delta)\|Qx\|_2 \leq \|AQx\|_2 \leq (1-\delta)\|Qx\|_2, \forall x \in \Sigma_K \tag{17}$$

with probability

$$\geq 1 - 2\left(1 + \frac{16 + 10\delta}{3\delta}\kappa(Q_K)\right)^k e^{-nc(\delta/4)} \tag{18}$$

where $\kappa(Q_K) = \frac{\sigma_{Q_K}}{\sigma_{Q_{Kmin}}}$

Proof. Since A satisfies the concentration inequality **??**, it must also satisfy the transformed concentration inequality by Corollary **??**. We denote E_i as the event such that $|\|AQu_i\|_2^2 - \|Qu_i\|_2^2| > \frac{\delta}{4}\|Qu_i\|_2^2$ for each u_i . We apply the transformed concentration inequality with $\epsilon = \delta/4$ and apply Boole's inequality to bound the "bad" events

$$Pr(\exists u_i, s.t, |\|AQu_i\|_2^2 - \|Qu_i\|_2^2| > \frac{\delta}{4}\|Qu_i\|_2^2) \leq \sum_{u_i \in \mathcal{U}} Pr(|\|AQu_i\|_2^2 - \|Qu_i\|_2^2| > \frac{\delta}{4}\|Qu_i\|_2^2) \leq 2|\mathcal{U}|e^{-nc(\frac{\delta}{4})}$$

$$\leq 2\left(1 + \frac{2\sigma_{Q_K}}{\xi}\right)^k e^{-nc(\frac{\delta}{4})}$$

Now suppose that the draw of matrix A gives

$$|\|AQu_i\|_2^2 - \|Qu_i\|_2^2| \leq \frac{\delta}{4}\|Qu_i\|_2^2 \Leftrightarrow (1-\frac{\delta}{4})\|Qu_i\|_2^2 \leq \|AQu_i\|_2^2 \leq (1+\frac{\delta}{4})\|Qu_i\|_2^2, \forall u_i \in \mathcal{U}$$

Which implies that

$$(1-\frac{\delta}{4})\|Qu_i\|_2 \leq \|AQu_i\|_2 \leq (1+\frac{\delta}{4})\|Qu_i\|_2, \forall u_i \in \mathcal{U}$$

because $\sqrt{1-\frac{\delta}{4}} > 1-\frac{\delta}{4}$ and $\sqrt{1+\frac{\delta}{4}} < 1+\frac{\delta}{4}$.

Consider δ' to be the smallest number such that

$$\|AQx\|_2 \leq (1+\delta')\|Qx\|_2, \forall x \in \Sigma_K$$

Given $x \in \Sigma_K$ with $\|x\|_2 = 1$, we can find a $u_i \in \mathcal{U} \subseteq \mathcal{S}_{\Sigma_K}$ such that

$$\|Qx - Qu_i\|_2 \leq \xi = \frac{3\delta}{8+5\delta}\sigma_{QKmin}$$

By triangular inequality,

$$|\|Qu_i\|_2 - \|Qx\|_2| \leq \|Qx - Qu_i\|_2 \leq \xi \Rightarrow \|Qx\|_2 - \xi \leq \|Qu_i\|_2 \leq \|Qx\|_2 + \xi$$

So that

$$\|AQx\|_2 \leq \|AQu_i\|_2 + \|A(Qx - Qu_i)\|_2 \leq \left(1+\frac{\delta}{4}\right)\|Qu_i\|_2 + (1+\delta')\|Qx - Qu_i\|_2$$

$$\leq \left(1+\frac{\delta}{4}\right)(\|Qx\|_2 + \xi) + (1+\delta')\xi$$

Since δ' to be the smallest number such that

$$\|AQx\|_2 \leq (1+\delta')\|Qx\|_2, \forall x \in \Sigma_K$$

So for any $x \in \Sigma_K$ we have

$$(1+\delta')\|Qx\|_2 \leq \left(1+\frac{\delta}{4}\right)(\|Qx\|_2 + \xi) + (1+\delta')\xi \Leftrightarrow (\delta' - \frac{\delta}{4})\|Qx\|_2 \leq \left(1+\frac{\delta}{4}+(1+\delta')\right)\xi$$

$$\leq \left(2+\frac{\delta}{4}+\delta'\right)\frac{3\delta}{8+5\delta}\sigma_{QKmin} \leq \left(2+\frac{\delta}{4}+\delta'\right)\frac{3\delta}{8+5\delta}\|Qx\|_2$$

The last inequality holds because $\|x\|_2 = 1$. So we have

$$(1+\delta') \leq \left(2+\frac{\delta}{4}+\delta'\right)\frac{3\delta}{8+5\delta}$$

A bit of algebra we can get $\delta' \leq \delta$, which means the following is always true

$$\|AQx\|_2 \leq (1+\delta)\|Qx\|_2, \forall x \in \Sigma_K \tag{19}$$

Now we consider the left hand side

$$\|AQx\|_2 \geq \|AQu_i\|_2 - \|A(Qx - Qu_i)\|_2 \geq \left(1-\frac{\delta}{4}\right)\|Qu_i\|_2 - (1+\delta')\|Qx - Qu_i\|_2$$

$$\geq \left(1-\frac{\delta}{4}\right)(\|Qx\|_2 - \xi) - (1+\delta')\xi = \left(1-\frac{\delta}{4}\right)\|Qx\|_2 - \xi\left(1-\frac{\delta}{4}+(1+\delta')\right)$$

the last inequality is because of the fact that $\|Qx\|_2 - \xi \leq \|Qu_i\|_2 \leq \|Qx\|_2 + \xi$. we continue,

$$\|AQx\|_2 \geq \left(1-\frac{\delta}{4}\right)\|Qx\|_2 - \xi\left(1-\frac{\delta}{4}+(1+\delta')\right) = \left(1-\frac{\delta}{4}\right)\|Qx\|_2 - \frac{3\delta}{8+5\delta}\sigma_{QKmin}\left(2-\frac{\delta}{4}+\delta'\right)$$

$$\geq \left(1-\frac{\delta}{4}\right)\|Qx\|_2 - \frac{3\delta}{8+5\delta}\|Qx\|_2\left(2-\frac{\delta}{4}+\delta'\right) \geq \left(1-\frac{\delta}{4}\right)\|Qx\|_2 - \frac{3\delta}{8+5\delta}\|Qx\|_2\left(2-\frac{\delta}{4}+\delta\right)$$

because of the fact that $\delta' \leq \delta$. We continue

$$\|AQx\|_2 \geq \left[\left(1-\frac{\delta}{4}\right) - \frac{3\delta}{8+5\delta}\left(2+\frac{3\delta}{4}\right)\right]\|Qx\|_2 \geq \left[\left(1-\frac{\delta}{4}\right) - \frac{3\delta}{4}\left(\frac{8+3\delta}{8+5\delta}\right)\right]\|Qx\|_2$$
$$\geq (1-\delta)\|Qx\|_2$$

which means that

$$(1-\delta)\|Qx\|_2 \leq \|AQx\|_2 \leq (1+\delta)\|Qx\|_2, \forall x \in \Sigma_K \tag{20}$$

$\square$

Theorem 2.1. *Suppose the matrix A satisfies concentration inequality* **??**. *Then for any $0 < \delta < 1$ and $n > c_1 k \ln\left(\frac{eN}{k}\right)$, there exist constants $c(\delta)$, $c_1(\kappa_{Q_k}, \delta)$ such that*

$$Pr(|\|AQx\|_2 - \|Qx\|_2| > \delta\|Qx\|_2) \leq 2e^{-c(\delta/4)(n - c_1 k \ln(eN/k))}$$

where

$$c_1 = \frac{1+\ln\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)}{c(\delta/4)}$$

Proof. For each k-dimensional subspace Σ_K, AQ will fail to satisfy Eqn **??** with probability

$$\leq 2\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_K)\right)^k e^{-nc(\delta/4)} \leq 2\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)^k e^{-nc(\delta/4)}$$

where $\kappa(Q_k)$ is the largest condition number of all possible sub-matrices Q_K. Now we allow the subset K to vary,

$$Pr(\exists x \in \Sigma_k, s.t, |\|AQx\|_2 - \|Qx\|_2| > \delta\|Qx\|_2) \leq \sum_{\Sigma_K \subseteq \Sigma_k} Pr(|\|AQx\|_2 - \|Qx\|_2| > \delta\|Qx\|_2)$$

$$\leq \binom{N}{k} \cdot 2\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)^k e^{-nc(\delta/4)} \leq 2\left(\frac{eN}{k}\right)^k \left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)^k e^{-nc(\delta/4)}$$
$$= 2e^{-nc(\delta/4)+k\left[\ln\left(\frac{eN}{k}\right)+\ln\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)\right]} \leq 2e^{-nc(\delta/4)+k\ln\left(\frac{eN}{k}\right)\left[1+\ln\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)\right]}$$

To ensure this value is small, we must impose

$$-nc(\delta/4) + k\ln\left(\frac{eN}{k}\right)\left[1+\ln\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)\right] < 0$$

Therefore,

$$n > c_1 k \ln\left(\frac{eN}{k}\right)$$

where

$$c_1 = \frac{1+\ln\left(1+\frac{16+10\delta}{3\delta}\kappa(Q_k)\right)}{c(\delta/4)}$$

$\square$

2.3 Transformed Isometry Property

Definition 2.1. *For each integer $k = 1, 2, ...,$ define the isometry constant δ_k of a matrix $B = AQ$ as the smallest number such that*

$$(1 - \delta_k)\|Qx\|_2^2 \le \|AQx\|_2^2 \le (1 + \delta_k)\|Qx\|_2^2, \forall x \in \Sigma_k \tag{21}$$

Note that conventional RIP is not equivalent to TIP, Qx is not a sparse vector in TIP so that we cannot let $y = Qx$ and use RIP directly. TIP holds if A satisfies concentration inequality with constant δ_k thus satisfies RIP with constant δ_k. From the above definition,we can deduce the *transformed approximate orthogonality*.

Lemma 2.3. *Denote $\sigma^{max}_{Q_{T\bigcup T'}}$ and $\sigma^{min}_{Q_{T\bigcup T'}}$ to be the maximum and minimum singular values of $Q_{T\bigcup T'}$. We have*

$$|\langle AQx, AQx' \rangle| \le \eta_{T,T'}\|x\|_2\|x'\|_2$$

where

$$\eta_{T,T'} = \frac{\sigma^{2(max)}_{Q_{T\bigcup T'}} - \sigma^{2(min)}_{Q_{T\bigcup T'}} + \delta_{k+k'}(\sigma^{2(max)}_{Q_{T\bigcup T'}} + \sigma^{2(min)}_{Q_{T\bigcup T'}})}{2}$$

for all x, x' supported on disjoint subsets $T, T' \subseteq 1, ..., N$ with $|T| \le k, |T'| \le k'$.

Proof. Let r and r' to be unit vectors such that $x = r\|x\|_2$ and $x' = r'\|x'\|_2$, we have

$$|\langle AQx, AQx' \rangle| = |\langle AQr, AQr' \rangle| \cdot \|x\|_2\|x'\|_2$$

Now we bound $|\langle AQr, AQr' \rangle|$. By transformed isometry property, $\forall r, r' \in \Sigma_k$

$$(1 - \delta_{k+k'})\|Q(r + r')\|_2^2 \le \|AQ(r + r')\|_2^2 \le (1 + \delta_{k+k'})\|Q(r + r')\|_2^2$$

$$(1 - \delta_{k+k'})\|Q(r - r')\|_2^2 \le \|AQ(r - r')\|_2^2 \le (1 + \delta_{k+k'})\|Q(r - r')\|_2^2$$

Suppose r is supported on T and r' is supported on T',

$$(1 - \delta_{k+k'})\|Q_{T\bigcup T'}(r + r')\|_2^2 \le \|AQ(r + r')\|_2^2 \le (1 + \delta_{k+k'})\|Q_{T\bigcup T'}(r + r')\|_2^2$$

$$(1 - \delta_{k+k'})\|Q_{T\bigcup T'}(r - r')\|_2^2 \le \|AQ(r - r')\|_2^2 \le (1 + \delta_{k+k'})\|Q_{T\bigcup T'}(r - r')\|_2^2$$

As r and r' are disjoint unit vectors,

$$\sqrt{2}\sigma^{min}_{Q_{T\bigcup T'}} = \sigma^{min}_{Q_{T\bigcup T'}}\|r \pm r'\|_2 \le \|Q_{T\bigcup T'}(r \pm r')\|_2 \le \sigma^{max}_{Q_{T\bigcup T'}}\|r \pm r'\|_2 = \sqrt{2}\sigma^{max}_{Q_{T\bigcup T'}}$$

So we have,

$$2(1 - \delta_{k+k'})\sigma^{2(min)}_{Q_{T\bigcup T'}} \le \|AQ(r \pm r')\|_2^2 \le 2(1 + \delta_{k+k'})\sigma^{2(max)}_{Q_{T\bigcup T'}}$$

By parallelogram identity,

$$|\langle AQr, AQr' \rangle| \le \frac{1}{4}|2(1+\delta_{k+k'})\sigma^{2(max)}_{Q_{T\bigcup T'}} - 2(1-\delta_{k+k'})\sigma^{2(min)}_{Q_{T\bigcup T'}}| = \frac{1}{2}[\sigma^{2(max)}_{Q_{T\bigcup T'}} - \sigma^{2(min)}_{Q_{T\bigcup T'}} + \delta_{k+k'}(\sigma^{2(max)}_{Q_{T\bigcup T'}} + \sigma^{2(min)}_{Q_{T\bigcup T'}})]$$

Therefore we have,

$$|\langle AQx, AQx' \rangle| = \frac{\|x\|_2\|x'\|_2}{2}[\sigma^{2(max)}_{Q_{T\bigcup T'}} - \sigma^{2(min)}_{Q_{T\bigcup T'}} + \delta_{k+k'}(\sigma^{2(max)}_{Q_{T\bigcup T'}} + \sigma^{2(min)}_{Q_{T\bigcup T'}})] = \eta_{T,T'}\|x\|_2\|x'\|_2$$

If we define $\sigma^{max}_{Q_{k+k'}} \in \sup\{\sigma^{max}_{Q_{T\bigcup T'}}, \forall T, T'\}$ and $\sigma^{min}_{Q_{k+k'}} \in \inf\{\sigma^{min}_{Q_{T\bigcup T'}}, \forall T, T'\}$, we have

$$\eta_{T,T'} \le \eta_{k+k'} = \frac{\sigma^{2(max)}_{Q_{k+k'}} - \sigma^{2(min)}_{Q_{k+k'}} + \delta_{k+k'}(\sigma^{2(max)}_{Q_{k+k'}} + \sigma^{2(min)}_{Q_{k+k'}})}{2}$$

□

2.4 TIP and Null-Space Property in ℓ_1

Lemma 2.4. *Let $B = AQ$ be any matrix which satisfies TIP of order $2k$ with*

$$\left(\frac{\sigma_{Q_{2k}}^{max}}{\sigma_{Q_{2k}}^{min}}\right)^2 \cdot \frac{1+\delta_{2k}}{1-\delta_{2k}} < \sqrt{2}+1$$

where $\sigma_{Q_{2k}}^{max} = \sup\{\sigma_{Q_{T\bigcup T'}}^{max}, \forall |T| = |T'| = k\}$ and $\sigma_{Q_{2k}}^{min} = \inf\{\sigma_{Q_{T\bigcup T'}}^{min}, |T| = |T'| = k\}$. Then B satisfy the null space property in ℓ_1.

Proof. Let $v \in \ker(B)$ and let T_0 be indices of the largest k entries of v, T_1 be indices of the next largest k entries of v, and so on. The last set T_s may have less than k elements. So we have

$$v = v_{T_0} + v_{T_1} + ... + v_{T_j} + ... + v_{T_s}$$

Since $v \in \ker(B)$, we have,

$$Bv_{T_0 \bigcup T_1} = -B(v_{T_2} + ... + v_{T_j} + ... + v_{T_s}) = -B\sum_{j\geq 2}^{s} v_{T_j}$$

We now bound $\|v_{T_0\bigcup T_1}\|_2$ by substituting the above equality,

$$\|Bv_{T_0\bigcup T_1}\|_2^2 = |\langle Bv_{T_0\bigcup T_1}, B\sum_{j\geq 2}^{s} v_{T_j}\rangle| = \sum_{j\geq 2}^{s}(|\langle Bv_{T_0}, Bv_{T_j}\rangle| + |\langle Bv_{T_1}, Bv_{T_j}\rangle|)$$

$$\leq \sum_{j\geq 2}^{s}(\eta_{T_0,T_j}\|v_{T_0}\|_2\|v_{T_j}\|_2 + \eta_{T_1,T_j}\|v_{T_1}\|_2\|v_{T_j}\|_2) \leq \eta_{2k}(\|v_{T_0}\|_2 + \|v_{T_1}\|_2)\sum_{j\geq 2}^{s}\|v_{T_j}\|_2$$

$$\leq \eta_{2k}(\|v_{T_0}\|_2 + \|v_{T_1}\|_2)\sum_{j\geq 2}^{s}\|v_{T_j}\|_2 \leq \sqrt{2}\eta_{2k}\|v_{T_0\bigcup T_1}\|_2\sum_{j\geq 2}^{s}\|v_{T_j}\|_2$$

We used the transformed approximate orthogonality proved in the previous section and the fact that $\|v_{T_0}\|_2 + \|v_{T_1}\|_2 \leq \sqrt{2}\|v_{T_0\bigcup T_1}\|_2$. Now we use the left hand side of TIP,

$$(1-\delta_{2k})\sigma_{Q_{2k}}^{2(min)}\|v_{T_0\bigcup T_1}\|_2^2 \leq (1-\delta_{2k})\sigma_{Q_{T_0\bigcup T_1}}^{2(min)}\|v_{T_0\bigcup T_1}\|_2^2 \leq \|Bv_{T_0\bigcup T_1}\|_2^2 \leq \sqrt{2}\eta_{2k}\|v_{T_0\bigcup T_1}\|_2\sum_{j\geq 2}^{s}\|v_{T_j}\|_2$$

Divide the inequality by $\|v_{T_0\bigcup T_1}\|_2$,

$$(1-\delta_{2k})\sigma_{Q_{2k}}^{2(min)}\|v_{T_0\bigcup T_1}\|_2 \leq \sqrt{2}\eta_{2k}\sum_{j\geq 2}^{s}\|v_{T_j}\|_2 \tag{22}$$

Now we try to bound $\sum_{j\geq 2}^{s}\|v_{T_j}\|_2$ by $\|v_{T_0^c}\|_1$ so that null space property in ℓ_1 can be established. It is intuitively true by norm equivalence, but we still want to achieve the tightest bound. For any $i \in T_j$ and $l \in T_{j-1}$, we have scalars $|v_i| \leq |v_l|$, so that, $|v_i| = \frac{k|v_i|}{k} \leq \frac{\|v_{T_{j-1}}\|_1}{k}$. Therefore by norm equivalence,

$$\|v_{T_j}\|_2 \leq \sqrt{k}\|v_{T_j}\|_\infty \leq \sqrt{k}\frac{\|v_{T_{j-1}}\|_1}{k} = \frac{\|v_{T_{j-1}}\|_1}{\sqrt{k}}$$

So we have,

$$\sum_{j\geq 2}^{s}\|v_{T_j}\|_2 \leq \sum_{j\geq 1}^{s}\frac{\|v_{T_j}\|_1}{\sqrt{k}} = \frac{1}{\sqrt{k}}\|v_{T_0^c}\|_1$$

substitute the above inequality into inequality **??**,

$$\|v_{T_0 \bigcup T_1}\|_2 \leq \frac{\sqrt{2}\eta_{2k}\|v_{T_0^c}\|_1}{\sqrt{k}(1-\delta_{2k})\sigma_{Q_{2k}}^{2(min)}} \tag{23}$$

Since $\|v_{T_0}\|_1 \leq \sqrt{k}\|v_{T_0}\|_2 \leq \sqrt{k}\|v_{T_0 \bigcup T_1}\|_2$, we have,

$$\|v_{T_0}\|_1 \leq \frac{\sqrt{2}\eta_{2k}\|v_{T_0^c}\|_1}{(1-\delta_{2k})\sigma_{Q_{2k}}^{2(min)}} \tag{24}$$

We let $\kappa_{Q_{2k}} = \sigma_{Q_{2k}}^{max}/\sigma_{Q_{2k}}^{min}$

$$\|v_{T_0}\|_1 \leq \frac{\sqrt{2}\eta'_{2k}}{(1-\delta_{2k})}\|v_{T_0^c}\|_1 \tag{25}$$

where

$$\eta'_{2k} = \frac{\kappa_{Q_{2k}}^2 - 1 + \delta_{2k}(\kappa_{Q_{2k}}^2 + 1)}{2}$$

ℓ_1-minimization procedure requires the coefficient of null space property in ℓ_1 to satisfy

$$\frac{\sqrt{2}\eta'_{2k}}{(1-\delta_{2k})} < 1$$

a little algebra gives,

$$\kappa_{Q_{2k}}^2 \cdot \frac{1+\delta_{2k}}{1-\delta_{2k}} < \sqrt{2}+1$$

If Q is canonical or orthonormal, i.e. $\kappa_{Q_{2k}} = 1$, straight away we have the condition $\delta_{2k} < \sqrt{2}-1$ which is the state-of-the-art RIP constant by Candes. This also verifies that our bound is tight. □

3 Diagonal dominant *f-LTM*

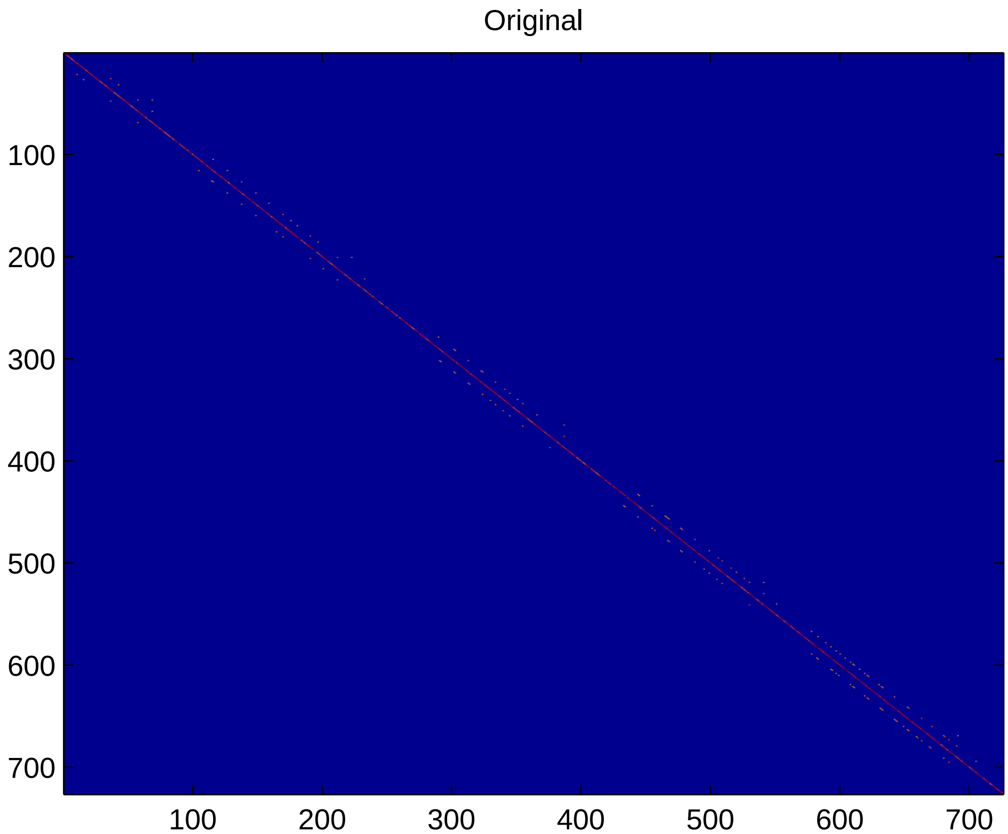

Figure 1: Ball-scene