Background Modelling on Tensor Field for Foreground Segmentation

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We propose a new method to background modeling using the tensor concept. The structure tensor was used to convert the image into a more information rich form and is defined as $\mathbf{T} = K_\rho * (vv^T)$ where $v = [I_x, I_y, I_r, I_g, I_b]$, and K_ρ is a smoothing kernel. The tensor space does not form a vector space, thus linear statistical techniques do not apply. Taking into account the differential geometrical properties of the Riemannian manifold where tensors lie [5], we propose a novel approach for foreground detection on tensor field based on data modeling by means of GMM directly on tensor domain. We introduced a K-means approximation of the EM algorithm based on an Affine-Invariant metric [5]. This metric has excellent theoretical properties but essentially due to the space curvature the computational burden is high. We propose a new K-means based on a new family of metrics, called Log-Euclidean [1], in order to speed up the process. Based on a novel vector space structure for tensors, the Log-Euclidean transforms computations on tensors into Euclidean computations on vectors in the logarithms domain. From a practical point of view yield similar results, with an experimental computation time ratio of at least 2 and sometime more in favor of the Log-Euclidean.

The tensor ($\mathbf{S}_d^+$ = $d \times d$ symmetric positive-definite matrix) space is not a vector space. Instead $\mathbf{S}_d^+$ lies on a Riemannian manifold M [2]. Let $\gamma(t) : [0,1] \to M$ be a curve, with $\gamma(0) = \mathbf{X}$, $\gamma(1) = \mathbf{Y}, \forall \mathbf{X},\mathbf{Y} \in M$. The *geodesic* between $\mathbf{X}$ and $\mathbf{Y}$ is defined as the minimum length curve connecting these points. The *tangent space* $T_\mathbf{X}M$ at $\mathbf{X}$ is the vector space which contains the *tangent vectors* to all curves on M passing through $\mathbf{X}$. Given a tangent vector $\dot\gamma(0) \in T_\mathbf{X}M$ there exists a unique geodesic with $\gamma(0) = \mathbf{X}$ and initial velocity $\dot\gamma(0)$. The *exponential map* $\exp_\mathbf{X} : T_\mathbf{X}M \to M$ maps the vector $\dot\gamma(0)$ at point $\mathbf{X}$ to $\mathbf{Y} = \gamma(1)$. The *logarithm map* $\log_\mathbf{X} : M \to T_\mathbf{X}M$ maps any point $\mathbf{Y} \in M$ to the unique tangent vector $\dot\gamma(0)$ at $\mathbf{X}$ that is the initial velocity of the geodesic from $\mathbf{X}$ to $\mathbf{Y}$.

Euclidean metric: the distance $\mathbf{D}_e(\mathbf{X},\mathbf{Y})$ between points $\mathbf{X},\mathbf{Y} \in \mathbf{S}_d^+$ and the gradient of the squared distance $\nabla_\mathbf{X}\mathbf{D}_e^2(\mathbf{X},\mathbf{Y})$ are given as follows

$$\left[\mathbf{D}_e(\mathbf{X},\mathbf{Y}) = |\mathbf{X}-\mathbf{Y}|_F = \sqrt{\mathrm{tr}((\mathbf{X}-\mathbf{Y})(\mathbf{X}-\mathbf{Y})^\mathbf{T})}\right] \quad \left[\nabla_\mathbf{X}\mathbf{D}_e^2(\mathbf{X},\mathbf{Y}) = \mathbf{X}-\mathbf{Y}\right] \quad (1)$$

Affine-Invariant metric [5] : the geodesic defined by the initial point $\gamma(0) = \mathbf{X}$ and the tangent vector $\dot\gamma(0)$ is expressed as

$$\gamma(t) = \exp_\mathbf{X}[t\dot\gamma(0)] = \mathbf{X}^{\frac{1}{2}} \exp\left[(t)\mathbf{X}^{-\frac{1}{2}}\,\dot\gamma(0)\mathbf{X}^{-\frac{1}{2}}\right]\mathbf{X}^{\frac{1}{2}} \quad (2)$$

which in case of $t = 1$ correspond to the exponential map $\exp_\mathbf{X} : \mathbf{T_X}M \to M$ with $\gamma(1) = \exp_\mathbf{X}(\dot\gamma(0))$. The logarithm map $\log_\mathbf{X} : M \to \mathbf{T_X}M$ is defined as

$$\dot\gamma(0) = \log_\mathbf{X}(\mathbf{Y}) = -\mathbf{X}\log(\mathbf{Y}^{-1}\mathbf{X}) \quad (3)$$

The geodesic distance $\mathbf{D}_a(\mathbf{X},\mathbf{Y})$ between points $\mathbf{X},\mathbf{Y} \in \mathbf{S}_d^+$, induced by this metric, derived from the Fisher information matrix [5] is given as

$$\left[\mathbf{D}_a(\mathbf{X},\mathbf{Y}) = \sqrt{\frac{1}{2}\mathrm{tr}(\log^2(\mathbf{X}^{-\frac{1}{2}}\mathbf{Y}\mathbf{X}^{-\frac{1}{2}}))}\right] \quad \left[\nabla_\mathbf{X}\mathbf{D}_a^2(\mathbf{X},\mathbf{Y}) = \mathbf{X}\log(\mathbf{Y}^{-1}\mathbf{X})\right] \quad (4)$$

The distance gradient, $\nabla_\mathbf{X}\mathbf{D}_a^2(\mathbf{X},\mathbf{Y})$ is the negative of the velocity $\dot\gamma(0)$.

Log-Euclidean metric [1] : based on specific properties of the matrix exponential/logarithm on tensors, it is possible to define a *vector space structure* on tensors. Since under the matrix exponentiation, there is a one-to-one mapping between the tensor space and the vector space of symmetric matrices, one can transfer to tensors the standard algebraic operations with the matrix exponential. The tensor vector space with this metric is in fact *isomorphic* and *isometric* with the corresponding Euclidean space of symmetric matrices. Results obtained on logarithms are mapped back to the tensor domain with the exponential. The geodesic defined by the point $\gamma(0) = \mathbf{X}$ and the tangent vector $\dot\gamma(0)$ is expressed as

$$\gamma(t) = \exp_\mathbf{X}[t\dot\gamma(0)] = \exp[\log(\mathbf{X}) + \partial_\mathbf{X}\mathbf{log.}[t\dot\gamma(0)]] \quad (5)$$

which in case of $t = 1$ correspond to the exponential map $\exp_\mathbf{X} : \mathbf{T_X}M \to M$ with $\gamma(1) = \exp_\mathbf{X}(\dot\gamma(0))$. The logarithm map $\log_\mathbf{X} : M \to \mathbf{T_X}M$ is defined as

$$\dot\gamma(0) = \log_\mathbf{X}(\mathbf{Y}) = \partial_{\log(\mathbf{X})}\mathbf{exp.}[\log(\mathbf{Y}) - \log(\mathbf{X})] \quad (6)$$

Since the Log-Euclidean metrics correspond to Euclidean metrics in the logarithms domain, the interpolation between tensors is simplified as

$$\gamma(t) = \exp[(1-t)\log(\mathbf{X}) + t\log(\mathbf{Y})] \quad (7)$$

The geodesic distance $\mathbf{D}_l(\mathbf{X},\mathbf{Y})$ between points $\mathbf{X},\mathbf{Y} \in \mathbf{S}_d^+$, induced by this metric, is also extremely simplified as follows

$$\left[D_l(\mathbf{X},\mathbf{Y}) = \sqrt{\mathrm{tr}[(\log(\mathbf{Y}) - \log(\mathbf{X}))^2]}\right] \quad \left[\nabla_\mathbf{X}D_l^2(\mathbf{X},\mathbf{Y}) = -\dot\gamma(0)\right] \quad (8)$$

The gradient, $\nabla_\mathbf{X}\mathbf{D}_l^2(\mathbf{X},\mathbf{Y})$ is the negative of the velocity $\dot\gamma(0)$. The Log-Euclidean D_l is much simpler than the Affine D_a where matrix multiplications, square roots, inverses are used. However, the mappings are complicated in the Log-Euclidean case by the use of the matrix differentials. Using spectral properties of symmetric matrices, one can compute an explicit/efficiently closed-form expression for these differentials [4].

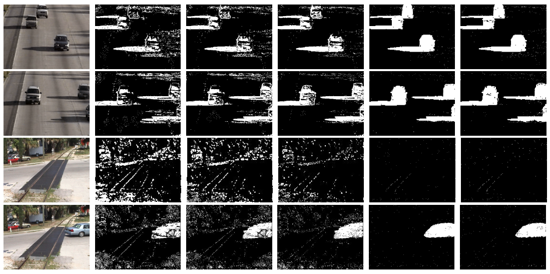

Figure 1: Left to right $\mapsto$ Original , GMM(v) , KDE(v) , GMM($\mathbf{T}$)-Euclidean , GMM($\mathbf{T}$)-AffineInvariant , GMM($\mathbf{T}$)-LogEuclidean

We model the background with a GMM on tensor space defined as

$$p(\mathbf{T}_i|\Theta) = \sum_{k=1}^{K}\omega_k \frac{\exp\left(-(1/2)\varphi(\beta_{i,k})^T\Lambda_k^{-1}\varphi(\beta_{i,k})\right)}{\sqrt{(2\pi)^n|\Lambda_k|}} \quad (9)$$

$\beta_{i,k} = -\nabla_{\bar{\mathbf{T}}_k}\mathbf{D}^2(\bar{\mathbf{T}}_k,\mathbf{T}_i)$, $\varphi : \mathbf{S}_d^+ \mapsto \Re^n$ is a *local coordinate chart*, ω_k=prior, $\bar{\mathbf{T}}_k$=mean, Λ_k=covariance. An EM algorithm as proposed in [3] for the Affine case is a costly procedure. In order to speed up the process we propose a online Kmeans, adapted from the version presented in [6].

Kmeans (Euclidean): the algorithm is similar to the [6]. The weights are updated using $\omega_k^t = (1-\alpha)\omega_k^{t-1} + (\alpha)(M_k^t)$, where M_k^t is 1 for the model which matched and 0 for the remaining models. The distribution parameters which matches the new observation ($\mathbf{T}_i$) are updated as follows

$$\left[\bar{\mathbf{T}}_k^t = (1-\rho)\bar{\mathbf{T}}_k^{t-1} + \rho\mathbf{T}_i\right] \quad \left[\Lambda_k^t = (1-\rho)\Lambda_k^{t-1} + \rho\,\varphi(\beta_{i,k}^t)^T\varphi(\beta_{i,k}^t)\right] \quad (10)$$

$$\left[\beta_{i,k}^t = -\nabla_{\bar{\mathbf{T}}_k^t}\mathbf{D}_e^2(\bar{\mathbf{T}}_k^t,\mathbf{T}_i) = \mathbf{T}_i - \bar{\mathbf{T}}_k^t\right] \quad \left[\rho = \alpha\mathcal{N}(\mathbf{T}_i|\bar{\mathbf{T}}_k^{t-1},\Lambda_k^{t-1})\right] \quad (11)$$

Kmeans (Affine-Invariant): the mean update equation (10) can only be directly applied in the Euclidean case. We need to take into account the Riemannian geometry of the manifold to apply the geodesic metrics. We propose a method to update the mean, based on the concept of tensor interpolation. In order to simplify we change the notation as follows

$$\left[\mathbf{Z} = \bar{\mathbf{T}}_k^t\right] \quad \left[\mathbf{X} = \bar{\mathbf{T}}_k^{t-1} = \gamma(0)\right] \quad \left[\mathbf{Y} = \mathbf{T}_i = \gamma(1)\right] \quad (12)$$

Let $\gamma(t) : [0,1] \subset \Re \to M$ be the geodesic defined by $\gamma(0) = \mathbf{X}$ and $\dot\gamma(0)$ with $\gamma(1) = \mathbf{Y}$. The point $\mathbf{Z}$ is the interpolation between $\mathbf{X}$ and $\mathbf{Y}$ at $t = \rho$

$$\mathbf{Z} = \gamma(\rho) = \mathbf{X}^{\frac{1}{2}}\exp\left[(\rho)\mathbf{X}^{-\frac{1}{2}}\left[-\mathbf{X}\log(\mathbf{Y}^{-1}\mathbf{X})\right]\mathbf{X}^{-\frac{1}{2}}\right]\mathbf{X}^{\frac{1}{2}} \quad (13)$$

$$\beta_{i,k}^t = -\nabla_{\bar{\mathbf{T}}_k^t}\mathbf{D}_a^2(\bar{\mathbf{T}}_k^t,\mathbf{T}_i) = -\nabla_\mathbf{Z}\mathbf{D}_a^2(\mathbf{Z},\mathbf{Y}) = -\mathbf{Z}\log(\mathbf{Y}^{-1}\mathbf{Z}) \quad (14)$$

Kmeans (Log-Euclidean): in this case, a closed-form expression for interpolation between tensors is given by (7). The point $\mathbf{Z}$ between $\mathbf{X}$ and $\mathbf{Y}$ that is reached by the geodesic $\gamma(t)$ at time $t = \rho$ is estimated as

$$\mathbf{Z} = \gamma(\rho) = \exp[(1-\rho)\log(\mathbf{X}) + \rho\log(\mathbf{Y})] \quad (15)$$

$$\beta_{i,k}^t = -\nabla_{\bar{\mathbf{T}}_k^t}\mathbf{D}_l^2(\bar{\mathbf{T}}_k^t,\mathbf{T}_i) = -\nabla_\mathbf{Z}\mathbf{D}_l^2(\mathbf{Z},\mathbf{Y}) = \partial_{\log(\mathbf{Z})}\mathbf{exp.}[\log(\mathbf{Y}) - \log(\mathbf{Z})] \quad (16)$$

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