Simple, Fast and Accurate Estimation of the Fundamental Matrix Using the Extended Eight-Point Schemes

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The eight-point scheme is the simplest and fastest scheme for estimating the fundamental matrix (FM) from a number $n \geq 8$ of noisy correspondences. As it ignores the fact that the FM must be singular, the resulting FM estimate is often inaccurate. Existing schemes that take the singularity constraint into consideration are several times slower and significantly more difficult to implement and understand. This paper describes extended versions of the eight-point (8P) and the weighted eight-point (W8P) schemes that effectively take the singularity constraint into consideration without sacrificing the efficiency and the simplicity of both schemes. The proposed schemes are respectively called the extended eight-point scheme (E8P) and the extended weighted eight-point scheme (EW8P). E8P minimizes an algebraic cost function whereas EW8P minimizes a weighted algebraic cost function.

The two proposed schemes have several advantages. First, they are as simple to understand and implement as the traditional 8P schemes. Second, the E8P scheme was experimentally found to give exactly the same results as Hartley's algebraic distance minimization scheme (ADMS) while being almost as fast as the simplest scheme (i.e., the 8P scheme). Third, the EW8P scheme not only improves the accuracy of the W8P scheme, but also runs about twice as fast. Experimental results indicate that it gives near-optimal results (its reprojection error is within 0.1% of the best result most of the time) while being 8-16 times faster than the more complicated schemes such as Levenberg-Marquardt schemes. Fourth, the EW8P scheme offers a very flexible structure that permits the use of geometric cost functions and/or robust weighting functions. Fifth, the FM estimates obtained by the E8P and the EW8P schemes perfectly satisfy the singularity constraint, eliminating the need for a post-processing step to enforce the rank-2 constraint.

Extended Eight-Point Scheme: The E8P scheme aims to minimize the same algebraic cost function of the 8P scheme but subject to the additional constraint that the estimated matrix is singular:

$$\text{argmin}_{\boldsymbol{f}} \; C_A(\boldsymbol{f}) = |\mathbf{M}\boldsymbol{f}|^2, \quad \text{subject to} \quad g_1(\boldsymbol{f}) = 0, \quad g_2(\boldsymbol{f}) = 0. \tag{1}$$

where $\boldsymbol{f}$ is the 9-vector containing the entries of the FM in row-major order, $\mathbf{M}$ is the $n \times 9$ measurement matrix [1], $g_1(\boldsymbol{f}) = |\boldsymbol{f}|^2 - 1$ and $g_2(\boldsymbol{f}) = \det \mathbf{F}(\boldsymbol{f})$. Introducing Lagrange multipliers converts (1) into the following unconstrained optimization problem:

$$\text{argmin}_{\boldsymbol{f},\lambda_1,\lambda_2} \; L(\boldsymbol{f},\lambda_1,\lambda_2) = |\mathbf{M}\boldsymbol{f}|^2 + 2\lambda_1 g_1(\boldsymbol{f}) + 2\lambda_2 g_2(\boldsymbol{f}). \tag{2}$$

At the optimal point $(\hat{\boldsymbol{f}},\lambda_1,\lambda_2)$, the partial derivatives ($\boldsymbol{\partial}_{\boldsymbol{f}} L$, $\boldsymbol{\partial}_{\lambda_1} L$, and $\boldsymbol{\partial}_{\lambda_2} L$) must vanish. This leads to the following system of equations:

$$\mathbf{M}^T\mathbf{M}\hat{\boldsymbol{f}} + \lambda_1 \boldsymbol{\partial}_{\boldsymbol{f}} g_1(\hat{\boldsymbol{f}}) + \lambda_2 \boldsymbol{\partial}_{\boldsymbol{f}} g_2(\hat{\boldsymbol{f}}) = \mathbf{0}, \quad g_1(\hat{\boldsymbol{f}}) = 0, \quad g_2(\hat{\boldsymbol{f}}) = 0. \tag{3}$$

Linearization: Assume that we are given a guess $\boldsymbol{f}_k$ of the minimizer $\hat{\boldsymbol{f}}$. Let ℓ_i be the first-order Taylor series approximation of the constraint function g_i at $\boldsymbol{f}_k$. It follows that ℓ_i has the following form:

$$\ell_i(\boldsymbol{f}) = g_i(\boldsymbol{f}_k) + \boldsymbol{\partial}_{\boldsymbol{f}}^T g_i(\boldsymbol{f}_k)(\boldsymbol{f} - \boldsymbol{f}_k). \tag{4}$$

If $\boldsymbol{f}_k$ is sufficiently close to $\hat{\boldsymbol{f}}$, ℓ_i well-approximates the behavior of g_i. Replacing g_i in Eq. (3) with ℓ_i and algebraically manipulating the resulting equations yield the following linear system of 11 equations:

$$\mathbf{M}^T\mathbf{M}\boldsymbol{f}_{k+1} + \mathbf{J}_k^T\boldsymbol{\lambda} = \mathbf{0}, \tag{5}$$

$$\mathbf{J}_k\boldsymbol{f}_{k+1} = \boldsymbol{c}_k. \tag{6}$$

where the 2×9 matrix $\mathbf{J}_k = \begin{bmatrix} \boldsymbol{\partial}_{\boldsymbol{f}} g_1(\boldsymbol{f}_k) & \boldsymbol{\partial}_{\boldsymbol{f}} g_2(\boldsymbol{f}_k) \end{bmatrix}^T$, the 2-vector $\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}^T$, and the 2-vector $\boldsymbol{c}_k = \mathbf{J}_k\boldsymbol{f}_k - (g_1(\boldsymbol{f}_k) \quad g_2(\boldsymbol{f}_k))^T$. This is a system of 11 equations in the vector of 11 unknowns $(\boldsymbol{f}_{k+1}^T \quad \boldsymbol{\lambda}^T)^T$. It can be solved using standard elimination techniques such as the LU decomposition.

The vector $\hat{\boldsymbol{f}}$ was replaced by $\boldsymbol{f}_{k+1}$ in Eq. (5) to indicate that solving this system will give us a better estimate $\boldsymbol{f}_{k+1}$ of the minimizer $\hat{\boldsymbol{f}}$ rather than the minimizer itself. $\boldsymbol{f}_{k+1}$ can then be used to obtain more accurate linearizations ℓ_i of the constraint functions g_i and recurrently solve the system of equations in (5) till convergence.

A key advantage of this scheme is that the count of flops it makes in each iteration is independent of n. Indeed, it takes only $(2d'^3/3)$ flops to solve a linear system such as (5) every iteration, where $d' = 11$. Yet, a more efficient update procedure can be obtained if the 9×9 moment matrix $\mathbf{A} = \mathbf{M}^T\mathbf{M}$ is invertible. In this procedure, the improved FM estimate $\boldsymbol{f}_{k+1}$ is obtained by evaluating the recurrent equation $\boldsymbol{f}_{k+1} = \mathbf{T}_k(\mathbf{N}_k^{-1}\boldsymbol{c}_k)$, where the 9×2 matrix $\mathbf{T}_k = \mathbf{A}^{-1}\mathbf{J}_k^T$ and the 2×2 symmetric matrix $\mathbf{N}_k = \mathbf{J}_k\mathbf{T}_k = \mathbf{J}_k\mathbf{A}^{-1}\mathbf{J}_k^T$. The derivation of this equation is given in the paper. This procedure further reduces the count of flops performed in each iteration to just $(2d^2)$, where $d = 9$.

The implementation described in the paper utilizes the SVD decomposition of the measurement matrix $\mathbf{M}$ to determine the rank of $\mathbf{A}$, obtain an initial estimate $\boldsymbol{f}_0$, and to calculate the inverse of $\mathbf{A}$. This implementation takes 3-4 iterations to converge in most of the cases.

Extended Weighted Eight-Point Scheme: The traditional W8P scheme attempts to minimize a cost function of the following form:

$$C(\boldsymbol{f}) = \sum_{i=1}^{n} (w_i \boldsymbol{m}_i^T \boldsymbol{f})^2, \quad \text{subject to } |\boldsymbol{f}|^2 = 1. \tag{7}$$

where each weight w_i depends on both the correspondence $\boldsymbol{M}_i$ and the FM $\boldsymbol{f}$. Similar to 8P, W8P can be extended to directly incorporate the rank-2 constraint into the optimization. The update equation in this case has the following form:

$$\begin{bmatrix} \mathbf{M}_k^T\mathbf{M}_k & \mathbf{J}_k^T \\ \mathbf{J}_k & \mathbf{0}_{2\times 2} \end{bmatrix} \begin{pmatrix} \boldsymbol{f}_{k+1} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{9\times 1} \\ \boldsymbol{c}_k \end{pmatrix}. \tag{8}$$

where the $n \times 9$ matrix $\mathbf{M}_k = \mathbf{W}_k\mathbf{M}$ and $\mathbf{W}_k$ is the $n \times n$ diagonal matrix $\text{diag}(w_1, w_2, \ldots, w_n)$. The paper describes how to select the weights w_i in order to make EW8P minimize geometric and robust cost functions.

Unlike the E8P scheme, the count of flops made in each iteration of the EW8P scheme is no more independent of n. This is due to the need to calculate a set of different weights and subsequently calculate a new moment matrix $\mathbf{A}_k = \mathbf{M}_k^T\mathbf{M}_k$ every iteration. This scheme may not, consequently, be as efficient as the E8P scheme. Practical experiments, however, show that the EW8P scheme runs many times faster than the other more complicated schemes (such as FNS, EFNS, LM). It is even faster than the less-accurate W8P scheme and Hartley's ADMS (as long as n is not very large).

Experiments: The proposed schemes were tested against several other existing schemes using pairs of real images with varying numbers of correspondences. Besides illustrating the accuracy and the efficiency of the proposed schemes, the experiments showed that FNS [2] and EFNS [2] diverge at some data sets.

[1] R. Hartley. Minimizing algebraic error in geometric estimation problems. In *ICCV*, pages 469–476, 1998.

[2] Y. Sugaya and K. Kanatani. Highest accuracy fundamental matrix computation. In *ACCV*, pages 311–321, 2007.

[3] PHS Torr and DW Murray. The development and comparison of robust methods for estimating the fundamental matrix. *IJCV*, 24(3): 271–300, 1997.