Diffusion-based Regularisation Strategies for Variational Level Set Segmentation

Maximilian Baust
baust@in.tum.de

Darko Zikic
zikic@cs.tum.edu

Nassir Navab
navab@cs.tum.edu

Computer Aided Medical Procedures
Technische Universität München
Munich, GER

Introduction Most variational level set methods for image segmentation can be summarized in the following recipe. First, design an energy E which gets minimized by the optimal configuration of the embedding function ϕ:

$$\min_{\phi \in V} E(\phi). \tag{1}$$

Second, apply the calculus of variations to obtain $\nabla E(\phi)$:

$$\frac{d}{ds} E(\phi + s\psi)\bigg|_{s=0} = \int_{\Omega} \nabla E(\phi) \cdot \psi \, dx = \langle \nabla E(\phi), \psi \rangle_{L^2} = 0. \tag{2}$$

Third, solve (1) via gradient descent, which leads to the continuous evolution equation

$$\partial_t \phi = -\nabla E(\phi) \tag{3}$$

and the discrete update equation

$$\phi^{t+\tau} = \phi^t - \tau \nabla E(\phi^t). \tag{4}$$

As already indicated in (2), the recipe described above implicitly assumes that $V = L^2$. Therefore it is mathematically correct to refer to ∇E as the L^2-gradient and not simply *the gradient*. Unfortunately, the L^2-gradient is, to put it simply, too local and therefore prone to lead into an undesired local minimum as pointed out by Charpiat *et al.* [2] as well as Sundaramoorthi *et al.* [3]. Thus regularisation strategies are necessary to avoid these undesired local minima.

In general, there are *implicit* regularisation strategies, which result in the choice of a smooth function space $V \subset L^2$ or *explicit* ones, which aim at minimizing a regularized energy:

$$\min_{\phi \in L^2} E(\phi) + \lambda R(\phi), \tag{5}$$

where R is an additional regularisation term. In this paper we propose diffusion-based regularisation strategies, which correspond to

$$R(\phi) = \int_{\Omega} (\nabla \phi)^T g \nabla \phi \, dx, \tag{6}$$

and compare them to the recently proposed ones of Charpiat *et al.* [2] and Sundaramoorthi *et al.* [3].

Regularization Paradigms We show that the implicit regularisation strategies of [2] and [3] result in update equations of the form

$$\phi^{t+\tau} = \phi^t - \tau \mathscr{R}\left[\nabla E(\phi^t)\right], \tag{7}$$

where $\mathscr{R}[\cdot]$ is either the *Gaussian operator*

$$\mathscr{G}(\sigma)[\psi] = G_\sigma * \psi, \tag{8}$$

or the *isotropic Sobolev operator*

$$\mathscr{S}(\alpha)[\psi] = (I - \alpha\Delta)^{-1} \psi. \tag{9}$$

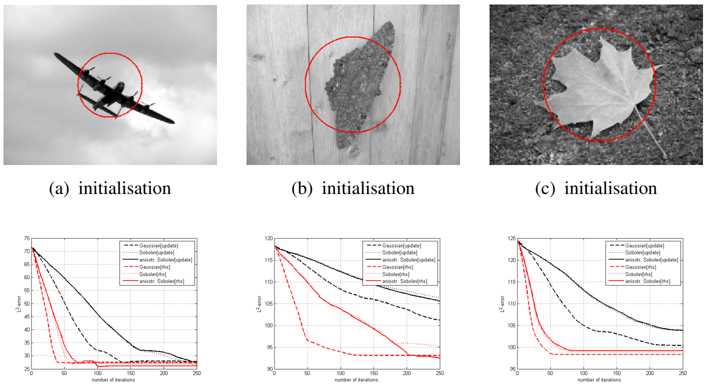
In addition to that, we introduce the *anisotropic Sobolev operator*

$$\mathscr{A}(\alpha)[\psi] = (I - \alpha \operatorname{div}(g\nabla))^{-1} \psi \tag{10}$$

and show that a diffusion-based regularisation results in

$$\phi^{t+\tau} = \mathscr{R}\left[\phi^t - \tau \nabla E(\phi^t)\right]. \tag{11}$$

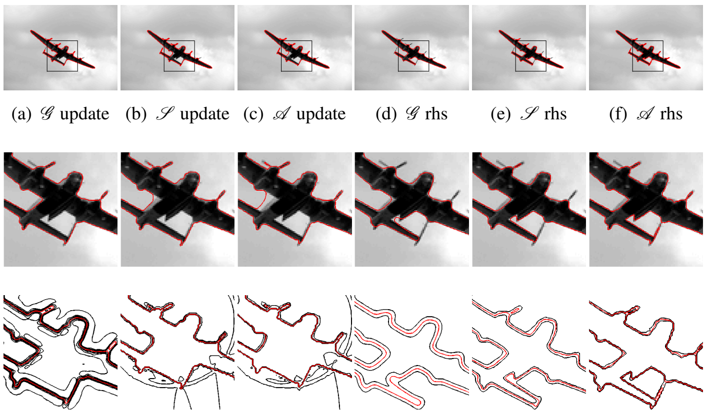
Thus we end up with two general regularisation paradigms, which are of the same computational complexity. We can apply a regularisation operator either to the update $\nabla E(\phi^t)$, or to the whole right-hand side (rhs) $\phi^t - \tau \cdot \nabla E(\phi^t)$, which corresponds to a diffusion regularization.



(a) initialisation (b) initialisation (c) initialisation

(d) convergence rates (e) convergence rates (f) convergence rates

Figure 1: The diffusion-based regularisation paradigm shows an increased convergence rate (images taken from [1]).



(a) $\mathscr{G}$ update (b) $\mathscr{S}$ update (c) $\mathscr{A}$ update (d) $\mathscr{G}$ rhs (e) $\mathscr{S}$ rhs (f) $\mathscr{A}$ rhs

Figure 2: Segmentation results (first row), close-ups of the results (second row), and the corresponding contour plots (bottom row).

Results A general observation is that it is always advisable to use a diffusion-based regularisation. In all three cases this results in an increased convergence rate (c.f. Fig. 1(d), 1(e), and 1(f)) and a smoother embedding function. Moreover, if we compare the results in Fig. 2, the regularisation of the whole right hand side seems to be less prone to get stuck in a local minimum of E.

Comparing the three regularisation operators with each other, it turns out that the Sobolev operator $\mathscr{S}(\alpha)$ enjoys the best compromise between runtime and quality: on the one hand, it can be implemented via a convolution with its impulse response $\mathscr{S}(\alpha)\delta$, which allows for a short runtime, and on the other hand, the quality of the results is visually satisfying.

[1] S. Alpert, M. Galun, R. Basri, and A. Brandt. Image segmentation by probabilistic bottom-up aggregation and cue integration. In *Computer Vision and Pattern Recognition, 2007. CVPR '07. IEEE Conference on*, pages 1 –8, 17-22 2007. doi: 10.1109/CVPR.2007.383017.

[2] G. Charpiat, P. Maurel, J.-P. Pons, R. Keriven, and O. Faugeras. Generalized gradients: Priors on minimization flows. *Int. J. Comput. Vision*, 73(3):325–344, 2007.

[3] Ganesh Sundaramoorthi, Anthony J. Yezzi, and Andrea Mennucci. Sobolev active contours. *International Journal of Computer Vision*, 73(3):345–366, 2007.