Revisiting Horn and Schunck: Interpretation as Gauss-Newton Optimisation

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In this paper we revisit the Horn and Schunck optical flow method [1], and focus on its interpretation as Gauss-Newton optimisation. We explicitly demonstrate that the standard incremental version of the Horn and Schunck (HS) method is equivalent to Gauss-Newton (GN) optimisation of the *non-linearised energy*, consisting of the sum of squared differences (SSD) criterion and diffusion regularisation. The formulation of incremental Horn and Schunck as Gauss-Newton optimisation has the following advantages.

- The proposed interpretation reveals that the incremental HS minimises a non-linearised energy. This affects one of the major points of criticism, which is that HS is applicable only for small displacements, since it minimises a linearised energy. This common misapprehension is presumably caused since many works based on HS do not state explicitly the overall energy, but only the linearised approximations of the energy, which are used in every iteration.
- Much simpler formulation and derivation of the HS method become possible please compare Figs. 1 and 2.
- Embedding the formulation in the well-understood non-linear least-squares (NLSQ) optimisation framework provides insight into the behaviour of the method.

It is interesting to note that although there are several scattered references of the relation between incremental HS and Gauss-Newton optimisation in the literature (cf. e.g. [2]), this interpretation is not widely spread and not the prevalent view in the community. The intention of this article is to emphasise on this interpretation, in contrast to previous works which had a different focus and were mentioning this relation mostly en passant. To this end, the equivalence between incremental HS and GN optimisation is explicitly demonstrated, and then, the implications of this interpretation are discussed in detail.

Beyond the above points, which we consider as the major contribution of this work, we also discuss the following topics.

- We analyse the effect of GN on variational motion estimation using SSD, compared to methods based on steepest gradient descent. The observation is that the characteristic effect of the GN scheme is that the local displacement updates experience a rescaling to approximately similar magnitudes when the GN optimisation scheme is applied. This results in assigning approximately the same weight to contributions from the complete image domain, independent on the local gradient strength (Fig. 3). It is this effect which is responsible for the high convergence rates of GN in low-gradient regions, compared to methods based on steepest gradient descent.
- The interpretation of the HS method as GN allows us to specify the class of difference measures which can be efficiently treated in the Horn and Schunck framework as the ones which feature a sparse Jacobian of the error term. This is the case for the so-called constancy assumptions, but in general not for statistical difference measures commonly used for multi-modal registration of medical images (e.g. Correlation Ratio, or Mutual Information).
- As examples for practical applications of the proposed analysis, we
 perform several variations on the theme of Horn and Schunck. The
 variations are theoretically justified by identifying GN as a specific
 preconditioning technique, and then modifying the form of the preconditioner. We present a strategy with a similar effect as GN, but
 applicable to arbitrary difference measures, and a further scheme,

From the Brightness Constancy Assumption

$$I(x + u(x), t) = I(x, t + \tau) ,$$

and its linearisation by Taylor expansion

$$I(x,t) + \frac{\partial I(x,t)}{\partial u}^{\mathsf{T}} u(x) = I(x,t) + \frac{\partial I(x,t)}{\partial \tau} \tau$$
,

one arrives at the Optical Flow Constraint (OFC)

$$0 = -\frac{\partial I(x,t)}{\partial u}^{\top} u(x) + \frac{\partial I(x,t)}{\partial \tau} \tau \ .$$

With $\tau = 1$, $I(x,t) = I_S(x)$, $I(x,t+1) = I_T(x)$, $\partial I/\partial u = \nabla I$, we get

$$0 = -\nabla I_{S}(x)^{\top} u(x) + (I_{T}(x) - I_{S}(x)) ,$$

resulting in the difference term

$$\frac{1}{2} \int_{\Omega} \left[I_{\mathrm{T}}(x) - I_{\mathrm{S}}(x) - \nabla I_{\mathrm{S}}(x)^{\top} u(x) \right]^{2} \mathrm{d}x ,$$

which together with the regularisation term yields the energy

$$\frac{1}{2} \int_{\Omega} \! \left[I_{\mathrm{T}}(x) \! - \! I_{\mathrm{S}}(x) \! - \! \nabla I_{\mathrm{S}}(x)^{\top} \! u(x) \right]^{2} \! \mathrm{d}x + \lambda^{2} \frac{1}{2} \int_{\Omega} \! \| \nabla u_{x}(x) \|^{2} \! + \! \| \nabla u_{y}(x) \|^{2} \! \mathrm{d}x \, ,$$

to be minimised in every iteration, followed by successive warping

$$I_{\rm S} \leftarrow I_{\rm S} \circ ({\rm id} + u)$$
.

Figure 1: Traditional derivation of incremental Horn and Schunck.

Minimise the energy

$$\frac{1}{2} \int_{\Omega} \left(I_{\mathrm{T}}(x) - I_{\mathrm{S}}(x + u(x)) \right)^{2} \mathrm{d}x + \lambda^{2} \frac{1}{2} \int_{\Omega} \|\nabla u_{x}(x)\|^{2} + \|\nabla u_{y}(x)\|^{2} \mathrm{d}x$$

by the method of Gauss-Newton.

Figure 2: Proposed formulation of incremental Horn and Schunck.







(a) Source whale image

(b) Update h for GN

(c) h for semi-implicit

Figure 3: Characteristic of Gauss-Newton for motion estimation: The magnitude of the local updates (colour saturation) depends far less on the corresponding local image gradients.

which results in decoupled linear systems. Furthermore, the NLSQ framework offers a simple justification for replacing the additional by the compositional update mode.

- [1] B.K.P. Horn and B.G. Schunck. Determining optical flow. Technical report, Massachusetts Institute of Technology, 1980.
- [2] J. Konrad. Use of colour in gradient-based estimation of dense twodimensional motion. In *Computer Vision: Systems, Theory and Ap*plications, 1993.

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¹While the original formulation of the HS method was not incremental, it was soon after its publication superseded by incremental versions in which the source image is warped in every iteration by the current estimate. This incremental version is treated as standard nowadays, and the original HS formulation is equivalent to the first iteration in this incremental process.