Multi-View Geometry of the Refractive Plane

Visesh Chari
http://perception.inrialpes.fr/~chari
Peter Sturm
http://perception.inrialpes.fr/people/Sturm/

INRIA Rhône Alpes & Laboratoire Jean Kuntzmann, Grenoble, France.

Transparent refractive objects are one of the main problems in geometric vision that have been largely unexplored. The imaging and multi-view geometry of scenes with transparent or translucent objects with refractive properties is relatively less well understood than for opaque objects. The main objective of our work is to analyze the underlying multi-view relationships between cameras, when the scene being viewed contains a single refractive planar surface separating two different media. Such a situation might occur in scenarios like underwater photography [2]. Our main result is to show the existence of geometric entities like the fundamental matrix, and the homography matrix in such instances. In addition, under special circumstances we also show how to compute the relative pose between two cameras immersed in one of the two media.

Projection Matrix The approach we take is along the lines of [1]. Representing a line by its Plücker coordinates, we first *back-project* a point to get a line $\mathbf{L}$. Then we extract the projection of an arbitrary line $\mathbf{L}_1$ onto the camera by analyzing the condition for the intersection of $\mathbf{L}$ and $\mathbf{L}_1$. As it turns out, a line $\mathbf{L}_1$ projects onto the image after refraction, as a quartic curve. This is expressed using a projection matrix, that relates the *lifted* Plücker coordinates of the line $\mathbf{L}_1$ to the coefficients of the quartic curve, represented by $\mathbf{c}$.

$$\mathbf{c} = \mathsf{P}\begin{pmatrix}\widehat{\mathbf{L}}_{1,(6,1,2)} \\ \widehat{\mathbf{L}}_{1,(4,5,3)}\end{pmatrix} \tag{1}$$

In general the matrix P is of dimensions 15×15, whose elements are bi-quadratic in terms of the external parameters of the camera, and a quadratic function of the relative refractive index.

Fundamental Matrix In deriving the projection matrix, we computed the projection of an arbitrary 3D line onto the image. We derive the fundamental matrix by replacing this line with the back-projection of an image point from the second camera. This gives us a matrix of dimensions 15×15 that relates the lifted Plücker coordinates of the image points of the two cameras as follows

$$\begin{pmatrix}\widehat{\mathbf{q}_2} & \widehat{\mathbf{q}_2} q_{2,3}^2\end{pmatrix} \mathsf{F} \begin{pmatrix}\widehat{\mathbf{q}} \\ \widehat{\mathbf{q}} q_3^2\end{pmatrix} = 0 \tag{2}$$

where $(\mathbf{q}_2, \mathbf{q})$ represent rotation-normalized image coordinates in the two images. The fundamental matrix has elements that are quartic in terms of the relative refractive index and bi-quadratic in the external parameters of the two cameras.

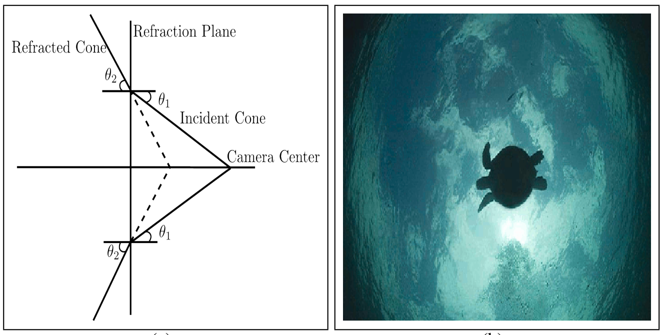
Homography In the case of homography relating a 3D plane to its image, a simpler representation of the transformation can be derived when the system of rays in consideration is parametrized using the incident and refracted angles (Figure 1(a)). Observe that all the refracted rays sharing the same angle of refraction converge to a single point (which is different for different angle values) even in 3D. Thus specific set of conics on a scene plane project onto the image, as conics. We express this relationship as a matrix that relates a point on the image to a point on the scene plane.

This is expressed in the equation for the homography matrix

$$\mathbf{S} \sim \mathbf{H}_{\theta_1}\mathbf{x} \tag{3}$$

where θ_1 is the incident angle, $\mathbf{x}$ is the image point and $\mathbf{S}$ is the corresponding point on the scene plane.

This matrix is of dimensions 4×3 and is a function of the normals of the refractive plane, the scene plane, the relative refractive index, the incident and refracted angles and the external parameters of the camera.



(a) (b)

Figure 1: (a) shows an illustration of refraction. The refracted lines, back-project to meet at several distinct *virtual* points, even in 3D. These lines cut by the image plane form conics.(b) shows an image of "Snell's Window", a conic that represents the horizon of the outside world. Here the conic is observed at the periphery of the image, beyond which the image blacks out due to total internal reflection. Photo courtesy gerb's photo-stream, http://www.flickr.com/photos/gerb/196296131/

Snell's Window In the specific case when the cameras are immersed in a denser medium than the scene, an interesting phenomenon known as the Snell's Window, is observed (Figure 1(b). The unique property of this window is that its equation in the image reveals information about both the refractive plane as well as the refractive index. In particular, the equation relating the conic to camera and refraction parameters is given by

$$\mathbf{x}^\top \left(\mathsf{R}^\top \mathbf{v}\mathbf{v}^\top \mathsf{R} - (1 - \tfrac{1}{\lambda^2})\mathsf{I}_{3\times 3}\right)\mathbf{x} = 0 \tag{4}$$

where $\mathbf{x}$ is the image coordinates, $\mathbf{v}$ is the refractive plane normal, R is a rotation matrix (camera external parameter) and λ is the relative refractive index. This shows that the periphery is a conic in image coordinates. The term $\mathsf{R}^\top\mathbf{v}$ represents the refractive plane normal in the camera coordinate system. One of the main advantages of Equation 4 is that one of the conic's eigenvectors is the normal of the refractive plane.

$$\left(\mathsf{R}^\top \mathbf{v}\mathbf{v}^\top \mathsf{R} - (1 - \tfrac{1}{\lambda^2})\mathsf{I}_{3\times 3}\right)\left(\mathsf{R}^\top \mathbf{v}\right) = \frac{1}{\lambda^2}\mathsf{R}^\top \mathbf{v} \tag{5}$$

since $\mathbf{v}^\top\mathbf{v} = 1$. In fact, its easy to show that the eigenvalues of the above matrix are $\frac{1}{\lambda^2}, -1 + \frac{1}{\lambda^2}, -1 + \frac{1}{\lambda^2}$, which means the only non-repeating eigenvalue gives information about the relative refractive index.

Relative Pose Finally, in the above mentioned scenario, it is also possible to solve for the relative pose when more than one camera is present. To do this, we observe that rotation and refractive index can be obtained by the above method. Rotation is only partially recovered. In all, for two cameras, 4 parameters comprising 3 translation and 1 rotation parameter remain to be recovered This can be done by using the Fundamental matrix equation and 4 correspondences.

Summary We have shown a number of interesting theoretical characteristics of planar refraction, and its relation to multiple view geometry. This helps us better understand situations like underwater vision with flat ports. We hope our analysis can provide useful benchmarking for structure recovery and other image analysis algorithms in such cases.

[1] P. Sturm and J. P. Barreto. General imaging geometry for central catadioptric cameras. In *ECCV 2008-Part II*, volume 4, pages 609–622, Marseille, France, 2008.

[2] T. Treibitz, Y. Y. Schechner, and H. Singh. Flat refractive geometry. In *CVPR 2008*, pages 1–8, Los Alamitos, CA, USA, 2008.