

Computation of Homographies

Matthew Harker and Paul O'Leary
Department of Product Engineering
University of Leoben
8700 Leoben, Austria
matthew.harker@stud.unileoben.ac.at
<http://automation.unileoben.ac.at>

Abstract

A new method for the non-iterative computation of a homography matrix is described. Rearrangement of the equations leads to a block partitioned sparse matrix, facilitating a residualization based on orthogonal matrix projections. This improves the handling of the error structure of the linear system of equations. The vanishing line is treated as the principal component in the estimation process. This estimate is more robust, since the position of the vanishing line depends only on the relative position and orientation of the camera to the observed plane, and is invariant to the structure of the points observed on the plane. A flop-count indicates that the new method is 11 times faster for four point correspondences, and converges to a factor of 5 for a large number of points.

Furthermore, a new non-iterative method of treating error in both images is derived. Combining the forward H and reverse G projections in a suitable manner eliminates the systematic bias of the estimation, and the first order error: a strict bound on the error reduction is derived. This can be achieved faster than a classical DLT due to the improved numerical efficiency. Results of Monte-Carlo simulations are presented to verify the performance.

1 Introduction

One procedure for the non-iterative computation of a homography, termed the direct linear transformation (DLT) is outlined in [6], and is common to much of the literature [9, 2]. The estimation is made by means of a total-least-squares (TLS) solution of a linear system of equations. This approach does not make much sense, as the error structure of the linear equations is not appropriate for TLS [3]. Mühlich et al. [7] attempt to overcome this through equilibration of the errors in variables. They propose to equilibrate only the Euclidean coordinates from one image in the estimation matrix. A proper equilibration should leave all columns with the same variance, and thus all columns must be appropriately equilibrated. The mixed columns of ones and zeros, have a variance when numerically computed, however, the columns are actually statistically invariant. The problem of poor error structure therefore remains.

We derive a methodology for the estimation of a homography matrix, based on the error structure of the estimation matrix. We quantitatively discuss the improvement in efficiency, and confirm the improvement in the estimation through numerical tests. The

proposed treatment is in fact general, and applies equally well to three-dimensional homographies, the estimation of a camera calibration matrix, amongst other problems with sparse block partitioned matrices such as coupled geometric objects [8].

2 Preliminaries

2.1 Derivation of the DLT Estimation

In two dimensions, the homography is a 3×3 matrix which maps homogeneous points as,

$$\mathbf{p}' = \mathbf{H}\mathbf{p} \quad \text{specifically,} \quad \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}. \quad (1)$$

A ubiquitous problem in computer vision is to estimate the homography matrix from a set of point correspondences. Of particular interest is the case where the point correspondences are between two images of the same plane. A linear algorithm can be derived by expanding Equation (1) for a given point correspondence, and normalizing with respect to the homogeneous component to yield,

$$x'_i = \frac{h_1x_i + h_2y_i + h_3}{h_7x_i + h_8y_i + h_9} \quad \text{and} \quad y'_i = \frac{h_4x_i + h_5y_i + h_6}{h_7x_i + h_8y_i + h_9}. \quad (2)$$

In this case, the point correspondences are assumed to be image coordinates, hence the “measured” homogeneous components are $w_i = w'_i = 1$. Rearranging the two equations leads to two equations that are linear in the elements of the homography, \mathbf{H} , i.e.

$$\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & -x'_ix_i & -x'_iy_i & -x'_i \end{bmatrix} \mathbf{h} = 0, \quad (3)$$

$$\begin{bmatrix} 0 & 0 & 0 & x_i & y_i & 1 & -y'_ix_i & -y'_iy_i & -y'_i \end{bmatrix} \mathbf{h} = 0, \quad (4)$$

hence, one point correspondence yields two equations. It is well known [6, 2, 7] that at least four point correspondences are required for a rank deficient 8×9 matrix, which is sufficient to solve for the nine parameters. For more than four point correspondences, a least-squares estimate must be made for the parameters. This methodology does not consider three issues critical to least-squares analysis: The uncertainty in measurements varies from column to column; the columns of constants that are statistically invariant; and the sparse nature of the system of equations. This leads to systematic errors in the least-squares technique which can be easily avoided.

2.2 The Generalized EYM Theorem

The EYM matrix approximation is generalized [3] for a matrix partitioned as,

$$\mathbf{A} = [\mathbf{A}_i \quad \mathbf{A}_j], \quad (5)$$

where $\text{rank} \mathbf{A}_j = l$. If the matrix \mathbf{A}_i is projected onto the orthogonal complement of \mathbf{A}_j , i.e. the space orthogonal to the column space of \mathbf{A}_j , then the resulting matrix, \mathbf{A}_i^\perp is orthogonal to \mathbf{A}_j . The matrix \mathbf{A}_i^\perp is thus the only portion of the matrix \mathbf{A}_i which contributes

to the rank of the overall matrix A greater than that of the sub-matrix A_j . Consequently, we perform the EYM approximation on A_i^\perp such that,

$$\hat{A}_i = \mathbf{H}_{r-l} \left(A_i^\perp \right), \quad (6)$$

where \mathbf{H}_{r-l} is an operator performing the rank $r-l$ EYM approximation, and recombine the rank r matrix \hat{A} as

$$\hat{A} = [\hat{A}_i \quad A_j]. \quad (7)$$

2.3 Data Preparation

It can be shown that the DLT algorithm is not invariant to arbitrary transformations of the data. This setback is overcome by normalization as described in [5, 1]. For each of the data sets, the centroid is subtracted, and the data is scaled such that the root-mean-square distance of the points to the origin is $\sqrt{2}$. The algorithm derived here assumes that this normalization has been performed, and the data sets are in fact mean-free.

3 Sparse Linear Systems and Orthogonalization

The approach we take here is to stack all the equations from Equation (3), then stack all the equations from Equation (4) below. This results in a system of equations which is clearly sparse in nature¹, i.e.

$$\mathbf{A}\mathbf{h} = \begin{bmatrix} \mathbf{P} & \mathbf{0} & \mathbf{X}'\mathbf{P} \\ \mathbf{0} & \mathbf{P} & \mathbf{Y}'\mathbf{P} \end{bmatrix} \mathbf{h} = \mathbf{r}, \quad (8)$$

where \mathbf{r} is the vector of algebraic residuals, i.e. the so-called algebraic distances. This arrangement is often overlooked in the literature, but as will be seen, holds several advantages over the standard methodology. In this case, the matrices are defined as

$$\mathbf{P} \triangleq [\hat{\mathbf{P}} \quad \mathbf{1}] \quad \text{with} \quad \hat{\mathbf{P}} \triangleq \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}, \quad (9)$$

as well as

$$\mathbf{X}' \triangleq \text{diag}(-x'_1, \dots, -x'_m) \quad \text{and} \quad \mathbf{Y}' \triangleq \text{diag}(-y'_1, \dots, -y'_m), \quad (10)$$

for ease of manipulation only; coding of the algorithm can be done more efficiently. When an exact solution, $\mathbf{r} = \mathbf{0}$, is not possible, the solution vector \mathbf{h} is given as the right singular vector corresponding to the smallest singular value of the matrix \mathbf{A} , and is the least-squares solution minimizing $\mathbf{r}^T \mathbf{r}$. Consider the definition of numerical rank, that is, r , such that

$$\sigma_1 \geq \dots \geq \sigma_r \geq \varepsilon \geq \sigma_{r+1} \geq \dots \geq \sigma_n \geq 0, \quad (11)$$

where σ_i are singular values and ε is the floating-point relative accuracy. The singular values and singular vectors are therefore directly related to the rank of the matrix in question. The sparse nature of the matrix \mathbf{A} gives us much more insight into this problem. Let

¹Specifically, one third of matrix \mathbf{A} is null

A_{i-j} denote a sub-matrix of A consisting of columns i through j . Consider the submatrix A_{1-6} , that is, the first six columns of the matrix A ,

$$A_{1-6} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}. \quad (12)$$

Clearly, we have, $A_{1-3} \perp A_{4-6}$, due to the zero sub-matrices. Furthermore, as per Section 2.3, it is assumed the data has undergone a normalization procedure, and is hence mean-free, that is, $\hat{P} \perp \mathbf{1}$. Consequently, we can say that

$$A_{1-2} \perp A_3 \perp A_{4-5} \perp A_6. \quad (13)$$

Since A_{1-2} is simply a permutation of the matrix A_{4-5} , we therefore have a simplified expression for the rank of the matrix A_{1-6} , that is,

$$\text{rank } A_{1-6} = 2 \text{rank } \hat{P} + 2. \quad (14)$$

The matrix \hat{P} is rank two as long as the points are not all collinear through the origin. Assuming that we have four point correspondences appropriate for computing a homography, then we can say that $\text{rank } A_{1-6} = 6$. As we are looking for rank deficiency, the matrix A_{1-6} is not appropriate as a principal component in the analysis. The expression for the rank of matrix A , given four appropriate point correspondences, is therefore

$$\text{rank } A = \text{rank } A_{7-9}^\perp + 6, \quad (15)$$

where A_{7-9}^\perp is the projection of the matrix A_{7-9} onto the orthogonal complement of A_{1-6} . Once again due to the sparse nature of matrix A , this orthogonal projection has a simplified form.

4 Partitioned Orthogonalization

4.1 Uncertainty in Measurements

To take full advantage of the sparse nature of A , we rewrite the system of equations as,

$$A_{1-6} \mathbf{h}_{1-6} + A_{7-9} \mathbf{h}_{7-9} = \mathbf{r}, \quad (16)$$

or,

$$\begin{bmatrix} \hat{P} & \mathbf{1} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \hat{P} & \mathbf{1} \end{bmatrix} \mathbf{h}_{1-6} + \begin{bmatrix} X' \hat{P} & X' \mathbf{1} \\ Y' \hat{P} & Y' \mathbf{1} \end{bmatrix} \mathbf{h}_{7-9} = \mathbf{r}. \quad (17)$$

Before proceeding with the orthogonalization, it is appropriate to discuss the elements of the matrix A in terms of measurements. The columns of ones are statistically invariant, that is, they are not actually measured values and do not contain any uncertainty in their values and hence have zero variance. Furthermore, they cannot be mean-free, and hence cause an unnecessary and erroneous bias to the TLS estimation. The columns which contain the matrix \hat{P} are measurements of the order of pixels in an image, and therefore contain a degree of uncertainty. And finally, the columns containing the matrix multiplications by X' and Y' represent the multiplication of two values both containing uncertainty, and therefore due to error propagation, they contain a higher degree of uncertainty. In the case of calibration, X' and Y' do not contain errors, which introduces additional invariance. On the whole, the matrix A derived by the DLT does not have an error structure suitable for TLS estimation.

4.2 Orthogonal Projections

The projection onto the orthogonal complement of a column of ones is equivalent to subtracting the mean values from the columns of the matrix. In this case, the only columns which are not mean-free are the first two in A_{7-9} . The system of equations is reduced to,

$$\begin{bmatrix} \hat{P} & 0 \\ 0 & \hat{P} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_4 \\ h_5 \end{bmatrix} + \begin{bmatrix} X' \hat{P} - \mathbf{1} \overline{X' \hat{P}} & X' \mathbf{1} \\ Y' \hat{P} - \mathbf{1} \overline{Y' \hat{P}} & Y' \mathbf{1} \end{bmatrix} \begin{bmatrix} h_7 \\ h_8 \\ h_9 \end{bmatrix} = \mathbf{r}' \quad (18)$$

with the corresponding values,

$$h_3 = -\overline{X' \hat{P}} \begin{bmatrix} h_7 \\ h_8 \end{bmatrix} \quad \text{and} \quad h_6 = -\overline{Y' \hat{P}} \begin{bmatrix} h_7 \\ h_8 \end{bmatrix}. \quad (19)$$

The notation used above denotes 1×2 matrices of second moments, i.e.

$$\overline{X' \hat{P}} = -\frac{1}{m} \begin{bmatrix} \sum_{i=1}^m x'_i x_i & \sum_{i=1}^m x'_i y_i \end{bmatrix} \quad \text{and} \quad \overline{Y' \hat{P}} = -\frac{1}{m} \begin{bmatrix} \sum_{i=1}^m y'_i x_i & \sum_{i=1}^m y'_i y_i \end{bmatrix}. \quad (20)$$

The projection onto the orthogonal complement is completed by orthogonalizing the matrix with respect to the block diagonal matrix containing \hat{P} and zero matrices 0. The reduced system is,

$$A_{7-9}^\perp \mathbf{h}_{7-9} = \begin{bmatrix} X' \hat{P} - \mathbf{1} \overline{X' \hat{P}} - \hat{P} \hat{P}^+ X' \hat{P} & X' \mathbf{1} - \hat{P} \hat{P}^+ X' \mathbf{1} \\ Y' \hat{P} - \mathbf{1} \overline{Y' \hat{P}} - \hat{P} \hat{P}^+ Y' \hat{P} & Y' \mathbf{1} - \hat{P} \hat{P}^+ Y' \mathbf{1} \end{bmatrix} \begin{bmatrix} h_7 \\ h_8 \\ h_9 \end{bmatrix} = \mathbf{r}'', \quad (21)$$

where \hat{P}^+ denotes the pseudo-inverse of the matrix \hat{P} given by $\hat{P}^+ = (\hat{P}^T \hat{P})^{-1} \hat{P}^T$. The relations for the eliminated coefficients are

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = -\hat{P}^+ X' \mathbf{P} \begin{bmatrix} h_7 \\ h_8 \\ h_9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h_4 \\ h_5 \end{bmatrix} = -\hat{P}^+ Y' \mathbf{P} \begin{bmatrix} h_7 \\ h_8 \\ h_9 \end{bmatrix}. \quad (22)$$

5 Constrained Minimization

Through the orthogonalization procedure, we have reduced the linear system to a function of only h_7 , h_8 , and h_9 , i.e.

$$A_{7-9}^\perp \mathbf{h}_{7-9} = \mathbf{r}'', \quad (23)$$

with backsubstitution relations for h_1 through h_6 given by Equations (19) and (22). To best approximate the null-space in terms of 2-norm distance, we take the minimizing solution to be the right singular vector of A_{7-9}^\perp corresponding to the smallest singular value². This implicitly imposes the constraint,

$$h_7^2 + h_8^2 + h_9^2 = 1. \quad (24)$$

²At this point, the matrix A_{7-9}^\perp should be equilibrated w.r.t. column variances, but due to space limitations this is not elaborated upon here; see [3].

The significance of this constraint is shown by reexamining Equation (1) where we have,

$$w' = h_7x + h_8y + h_9w. \quad (25)$$

For points $[x \ y \ 1]^T$ mapping to a point with $w' = 0$, we correspondingly have

$$h_7x + h_8y + h_9 = 0, \quad (26)$$

which is the equation of the vanishing line of the original image. The vanishing line is therefore treated as the principal component in the analysis. The position of the vanishing line depends on the position of the plane in question with respect to the camera plane, and not on the selection of points on that plane. This invariant nature of the vanishing line is further argument to treat it as the principal component, in addition to the fact that its coefficients in A_{7-9} comprise the greatest uncertainty.

Further constraints are imposed by the backsubstitution relations of Equations (19) and (22). These are based on pseudo-inverses, and are hence least-squares solutions for the coefficients h_1 through h_6 based on the estimate of h_7 , h_8 , and h_9 .

6 Computational Efficiency

The above procedure may seem very complicated, indeed the derivation is rather involved. The computational efficiency, however, is significantly better than that of the DLT. The proposed algorithm reduces the $2m \times 9$ linear system of the DLT to a $2m \times 3$ linear system. As both algorithms require singular value decomposition, we discuss algorithm efficiency in terms of flop-counts [4]. The new algorithm requires additional matrix computations due to the orthogonalization procedure; a tally for both algorithms is given in Table 1. Whereas both algorithms require a singular value decomposition, neither requires the information in the matrix U . We compare for two cases, one where U , S , and V are provided, and one where only S and V are provided, as the latter algorithm is more efficient, but not included in the MATLAB[®] package. The DLT algorithm has a large offset of 5832 flops;

Table 1: Flop-Counts for the Standard and Proposed Methods

Method	U , S , and V	S , and V
DLT	$2272m + 5832$	$652m + 5832$
Proposed Method	$302m + 241$	$122m + 241$

the proposed algorithm thus requires a factor of 11 times fewer flops for the minimum of four point correspondences. This converges to a factor of 5 improvement for arbitrarily large amounts of points.

7 Error in Both Images

One setback of the DLT, as well as the algorithm proposed here, is that they are biased to one set of points. If this were not the case, we might swap the places of points \mathbf{p} and \mathbf{p}' , compute the new transformation \mathbf{G} , which maps points as $\mathbf{p} = \mathbf{G}\mathbf{p}'$, and \mathbf{G} would be precisely the inverse of matrix \mathbf{H} , i.e. $\mathbf{H}\mathbf{G} = \mathbf{I}$. Unfortunately, this is not the case.

The proposed algorithm is efficient enough that both projections can be computed more efficiently than one with the DLT. In this case, we have H and G such that,

$$HG = I + \Delta, \quad (27)$$

where the error matrix Δ is the deviation of the above computation from the identity matrix I. We propose an interpolation scheme to minimize this error, where

$$\bar{H} = \mu H + \lambda \varepsilon G^{-1} \quad (28)$$

$$\bar{G} = \mu H^{-1} + \lambda \varepsilon G, \quad (29)$$

and $\varepsilon = \text{sign}(\det(HG))$ is necessary due to scaling³. The new error term is found by expanding the assumed solutions, i.e.

$$\begin{aligned} \bar{H}\bar{G} &= (\mu H + \lambda \varepsilon G^{-1})(\mu H^{-1} + \lambda \varepsilon G) \\ &= (\mu^2 + \lambda^2 \varepsilon^2) I + \mu \lambda \varepsilon (HG + (HG)^{-1}). \end{aligned} \quad (30)$$

But we have that $HG = I + \Delta$, and consequently the required inverse has the form,

$$(HG)^{-1} = I - \Delta(I + \Delta)^{-1}. \quad (31)$$

The expression in Equation (30) simplifies to

$$\bar{H}\bar{G} = (\mu + \lambda \varepsilon)^2 I + \mu \lambda \varepsilon (\Delta - \Delta(I + \Delta)^{-1}). \quad (32)$$

The question arises, to what degree does this interpolation annihilate the error in the approximation? The new error term can be considered to be

$$E \triangleq \mu \lambda \varepsilon (\Delta - \Delta(I + \Delta)^{-1}). \quad (33)$$

If we set a bound on the initial error in terms of an arbitrary p -norm, then we can quantify the new error in terms of the original error. A more than reasonable bound is $\|\Delta\|_p < 1$, considering that is generally the order of the estimation itself, H or G. Under this assumption [4], we can say that, $(I + \Delta)^{-1} = \sum_{k=0}^{\infty} \Delta^k$. Consequently, the error term is,

$$E = \mu \lambda \varepsilon \left(\Delta - \Delta \left(I + \sum_{k=1}^{\infty} \Delta^k \right) \right) \quad (34)$$

$$= \pm \mu \lambda \Delta \sum_{k=1}^{\infty} \Delta^k = \pm \mu \lambda \sum_{k=2}^{\infty} \Delta^k, \quad (35)$$

which essentially says that we have annihilated the first order error term. That is, if the second order terms can be neglected, then the error is negligible. A more quantitative assessment can be given in terms of bounds on the p -norm of the error term,

$$\|E\|_p = |\mu \lambda \varepsilon| \left\| \Delta \left(I - (I + \Delta)^{-1} \right) \right\|_p \leq |\mu \lambda| \|\Delta\|_p \left\| I - (I + \Delta)^{-1} \right\|_p \quad (36)$$

$$\leq |\mu \lambda| \frac{\|\Delta\|_p^2}{1 - \|\Delta\|_p}. \quad (37)$$

³Due to homogeneous scaling the identity matrix of Equation (27) may in fact be a negative multiple, thus $\varepsilon = \pm 1$.

We are therefore interested in the range where the original error is larger than the upper bound of the error after interpolation, i.e. when

$$\|\Delta\|_p \geq |\mu\lambda| \frac{\|\Delta\|_p^2}{1 - \|\Delta\|_p}. \quad (38)$$

Noting that matrix norms must be positive, this relation holds for $\|\Delta\|_p$ in the range,

$$0 \leq \|\Delta\|_p \leq \frac{1}{1 + |\mu\lambda|}. \quad (39)$$

If we constrain the coefficients μ and λ such that $\mu + \lambda = 1$ making the coefficient of the identity matrix in Equation (32) equal to 1, then product $\mu\lambda$ reaches a maximum value of $\frac{1}{4}$ over $0 \leq \mu \leq 1$. In this case, $E = \pm \frac{1}{4} \sum_{k=2}^{\infty} \Delta^k$, and the error is guaranteed to be reduced over the range

$$0 \leq \|\Delta\|_p \leq \frac{4}{5}. \quad (40)$$

Furthermore, the range over which the error is reduced by at least an order of magnitude, i.e. $\|E\|_p \leq 10^{-1} \|\Delta\|_p$, is found to be,

$$0 \leq \|\Delta\|_p \leq \frac{2}{7} \approx 0.2857. \quad (41)$$

As for selecting the parameters μ and λ , the choice of $\mu = \lambda = \frac{1}{2}$ appears to be ideal, as it weights the error in both images equally. The geometric significance of this is shown, as the matrix \bar{H} maps the point \mathbf{p} to the point \mathbf{p}' as,

$$\mathbf{p}' = \frac{1}{2} (\mathbf{H}\mathbf{p} + \varepsilon\mathbf{G}^{-1}\mathbf{p}), \quad (42)$$

which is the centroid of the two points transformed by \mathbf{H} and \mathbf{G}^{-1} individually.

8 Numerical Testing

We compared the DLT with the proposed algorithm through a Monte-Carlo simulation of the uncertainty of a reprojected point. The testing was performed on an image of a tiled floor and the ideal (error-free) floor plan in Figure 1. The point correspondences were determined manually, as precisely as possible (48 tile corner points). Gaussian noise with a standard deviation of one pixel was added to the corner points used to compute the homographies. The positions of three people (\mathbf{P}_1 , \mathbf{P}_2 , and \mathbf{P}_3) were mapped to the ideal frame, and back projected to the image using the estimated forward and back projections. The centroid of these three points was used to determine the statistical variance. A Monte-Carlo simulation of 1000 iterations is shown in Figure 2. The results show that the reprojection uncertainty of the proposed algorithm is considerably more compact than that of the DLT.

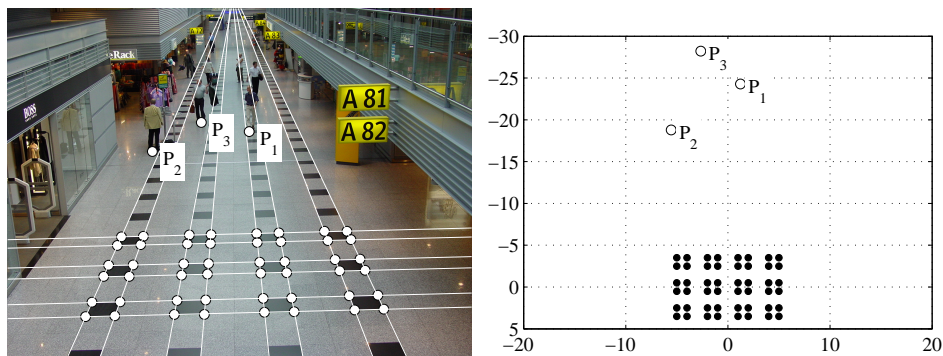


Figure 1: (Left) The image of a tiled floor with 48 corner points identified and (Right) the data rectified to an ideal model. Points P_1 , P_2 , and P_3 are the positions of the three foremost people.

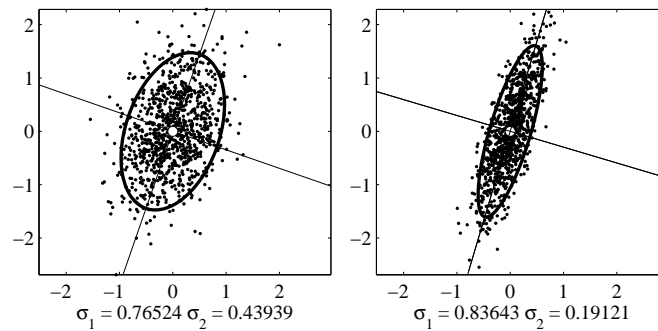


Figure 2: The scatters from a Monte-Carlo simulation of reprojection covariance for the DLT (Left) and the new method (Right). The ellipses represent two-standard-deviation confidence envelopes. The interpolation scheme was not used in this test.

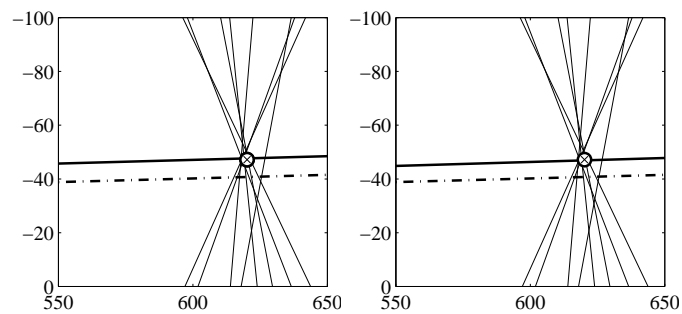


Figure 3: (Left) The vanishing point predicted by the tiled floor and the two vanishing lines predicted by the DLT (---) and the new algorithm (—). (Right) The vanishing lines predicted by the DLT (---) and the interpolation scheme (—).

Figure 2 shows clearly anisotropic error behaviour for both algorithms, whereby the standard deviation along the semi-minor axis is significantly reduced with the new algorithm. Furthermore, the estimation bias, i.e. the distance of the vanishing point to the vanishing line, is considerably reduced by the new algorithm; see Figure 3.

9 Conclusions

A new non-iterative method for the computation of a homography matrix from four or more point correspondences has been derived. In comparison to the standard DLT, the method was shown to be at least a factor of 5 times more efficient, and the uncertainty of error-prone reprojected points was shown to be considerably more compact. The factor of 11 improvement in efficiency for four point correspondences makes it more appropriate than the DLT for use in RANSAC homography estimation. The method derived is in fact general, and can, for example, be applied to the similarly sparse problems of three-dimensional homographies, and camera calibration, or problems with similar error structure, such as estimating the fundamental matrix.

References

- [1] W. Chojnacki, M. J. Brooks, A. van den Hengel, and D. Gawley. Revisiting Hartley's normalized eight-point algorithm. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 25(9):1172–1177, 2003.
- [2] A. Criminisi, I. Reid, and A. Zisserman. A plane measuring device. *Image and Vision Computing*, 17(8):625–634, 1999.
- [3] G.H. Golub, A. Hoffman, and G.W. Stewart. A generalization of the Eckart-Young-Mirsky matrix approximation theorem. *Linear Algebra and its Applications*, 88/89:317–327, 1987.
- [4] G.H. Golub and C.F. Van Loan. *Matrix Computations*. John Hopkins University Press, Baltimore, third edition, 1996.
- [5] R. Hartley. In defense of the eight-point algorithm. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 19(6):580–593, 1997.
- [6] R. Hartley and A. Zisserman. *Multiple View Geometry in Computer vision*. Cambridge University Press, Cambridge, 2001.
- [7] M. Mühlich and R. Mester. A considerable improvement in non-iterative homography estimation using TLS and equilibration. *Pattern Recognition Letters*, 22(11):1182–1189, 2001.
- [8] P. O'Leary, M. Harker, and P. Zsombor-Murray. Direct least-squares fitting of coupled geometric objects. *IEE Proc. Vision, Image and Signal Processing*, In Press, 2005.
- [9] G. Simon, A.W. Fitzgibbon, and A. Zisserman. Markerless tracking using planar structures in the scene. In *Proc. International Symposium on Augmented Reality*, pages 120–128, 2000.