Direct and Specific Fitting of Conics to Scattered Data

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Abstract

A new method to fit specific types of conics to scattered data points is introduced. Direct, specific fitting of ellipses and hyperbolæ is achieved by imposing a quadratic constraint on the conic coefficients, whereby an improved partitioning of the design matrix is devised so as to improve computational efficiency and numerical stability by eliminating redundant aspects of the fitting procedure. Fitting of parabolas is achieved by determining an orthogonal basis vector set in the Grassmannian space of quadratic conic forms. The linear combination of the basis vectors which fulfills the parabolic condition and has a minimum residual is determined using Lagrange multipliers.

1 Introduction

This paper addresses the problem of fitting a specific type of conic to scattered data, e.g. finding the best hyperbolic approximation to a set of data points. Solutions are provided for all three types of conic, i.e. hyperbolæ, ellipses and parabolas, together with their degenerate forms. This is especially useful when *a-priori* knowledge of the problem indicates the type of conic to be fit.

This problem was addressed by Nievergelt [10], however the quadratic constraint used by him leads to a general fit. The result of the fit is tested for its type; if it is not of the sought type then he proceeds to solve a *geodetic* equation leading to the nearest conic of the desired type. Quadratic constrained least squares was first successfully applied by Fitzgibbon et al. [3] to the problem of ellipse specific fitting — a task which was considered to be fundamentally non-linear up to that time. The work of Fitzgibbon et al. was extended by O'Leary et al. [11] to solve ellipse specific and hyperbola specific fitting. However, a parabola specific fit cannot be solved using standard quadratic constraints since it requires a zero constraint, which cannot be implemented by Lagrange multipliers. The most important contributions of this paper are:

1. A new linear parabola specific fitting method.
2. An improved matrix partitioning, extending the work of Halir et al. [7]. An incremental orthogonal residualization of the partitioned scatter matrix is performed which corresponds to a generalization of the Eckart-Young-Mirsky theorem [5].

The theoretical background to the proposed methods is presented and verified by comprehensive numerical testing.

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2 Data Preparation

Chojnacki et al. [2] proved that ensuring the data is mean free and scaled to have a root-mean-square distance of $\sqrt{2}$ to the origin improves the numerical performance and statistical behaviour of a fitting algorithm. In a planar fit this involves subtracting centroid coordinates $(\bar{x},\bar{y})$ from raw data (x_i,y_i) to give so-called *mean-free* coordinates $(\hat{x}_i,\hat{y}_i) = (x_i - \bar{x}, y_i - \bar{y})$. The appropriate scaling factor m imposes the metric, with $m = \sqrt{\frac{2n}{\sum_{i=1}^{n}(\hat{x}_i^2+\hat{y}_i^2)}}$. Therefore the data set becomes $(x_i,y_i) \triangleq (m\hat{x}_i, m\hat{y}_i)$.

3 Reduction of the Scatter Matrix

The notation for the quadratic forms in the projective plane, i.e. the conic sections, used in standard literature on geometry [8] is,

$$\mathbf{p}^T \mathsf{K} \mathbf{p} = \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0, \tag{1}$$

where K is the conic matrix, and $\mathbf{p}$ is a homogeneous point with w the homogeneous coordinate. This conic equation can be written as a product of vectors, i.e.

$$\mathbf{dz} = \begin{bmatrix} x^2 & xy & y^2 & x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T = 0. \tag{2}$$

The "design" $\mathbf{d}$ and coefficient $\mathbf{z}$ vectors are respectively the dual-Grassmannian and Grassmannian coordinates of the conics. Given n points, the vector $\mathbf{d}$ becomes the n-row design matrix D. This results in a vector $\mathbf{r}$ which is the residual vector of the n points in the conic equation whose norm is to be minimized, and corresponds to the algebraic distances of the points to the conic. The partitioning of the design matrix D and coefficient vector $\mathbf{z}$ is proposed as follows,

$$\mathsf{D}\mathbf{z} = \mathbf{r} = \begin{bmatrix} \mathsf{D}_2 & \mathsf{D}_1 & \mathsf{D}_0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{z}_1 \\ z_0 \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}. \tag{3}$$

In this case, the matrices are partitioned into groupings of their quadratic, linear, and constant terms, i.e.

$$\mathsf{D}_2 = \begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_ny_n & y_n^2 \end{bmatrix}, \quad \mathsf{D}_1 = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}, \quad \text{and} \quad \mathsf{D}_0 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{4}$$

and therefore,

$$\mathbf{z}_2 = \begin{bmatrix} a & b & c \end{bmatrix}^T, \quad \mathbf{z}_1 = \begin{bmatrix} d & e \end{bmatrix}^T \quad \text{and} \quad z_0 = f. \tag{5}$$

The sum of the squared residuals is thus,

$$\begin{bmatrix} \mathbf{z}_2^T & \mathbf{z}_1^T & z_0 \end{bmatrix} \begin{bmatrix} \mathsf{S}_{22} & \mathsf{S}_{21} & \mathsf{S}_{20} \\ \mathsf{S}_{12} & \mathsf{S}_{11} & \mathsf{S}_{10} \\ \mathsf{S}_{02} & \mathsf{S}_{01} & \mathsf{S}_{00} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{z}_1 \\ z_0 \end{bmatrix} = \mathbf{r}^T \mathbf{r}, \tag{6}$$

where the scatter matrices S_{ij} are defined as, $\mathsf{S}_{ij} \triangleq \mathsf{D}_i^T \mathsf{D}_j$, noting of course that $\mathsf{S}_{ij} = \mathsf{S}_{ji}^T$, and $\mathsf{S}_{00} = n$. The unique minimum of the least squares problem occurs when the set of partial derivatives of Equation 6 are equal to zero, i.e. when,

$$
\begin{aligned}
\mathsf{S}_{22}\mathbf{z}_2 + \mathsf{S}_{21}\mathbf{z}_1 + \mathsf{S}_{20}z_0 &= 0 &\quad (7)\\
\mathsf{S}_{21}^T\mathbf{z}_2 + \mathsf{S}_{11}\mathbf{z}_1 + \mathsf{S}_{10}z_0 &= 0 &\quad (8)\\
\mathsf{S}_{20}^T\mathbf{z}_2 + \mathsf{S}_{10}^T\mathbf{z}_1 + nz_0 &= 0. &\quad (9)
\end{aligned}
$$

The partial derivative with respect to z_0, Equation 9, implies that,

$$
z_0 = -\frac{1}{n}\begin{bmatrix}\mathsf{S}_{20}^T & \mathsf{S}_{10}^T\end{bmatrix}\begin{bmatrix}\mathbf{z}_2\\ \mathbf{z}_1\end{bmatrix} = -\begin{bmatrix}\overline{x^2} & \overline{xy} & \overline{y^2} & \overline{x} & \overline{y}\end{bmatrix}\begin{bmatrix}\mathbf{z}_2\\ \mathbf{z}_1\end{bmatrix}. \quad (10)
$$

This directly states that the linear fit, which in this space is a hyperplane, must pass through the centroid of the points in the corresponding space. Nievergelt [9] applies this to the fitting of hyperplanes and hyperspheres. Thus, with mean-free planar data, if we apply a transformation in the hyperspace such that,

$$
\mathsf{D}_2 \triangleq \begin{bmatrix} x_1^2 - \overline{x^2} & x_1y_1 - \overline{xy} & y_1^2 - \overline{y^2} \\ \vdots & \vdots & \vdots \\ x_n^2 - \overline{x^2} & x_ny_n - \overline{xy} & y_n^2 - \overline{y^2} \end{bmatrix}, \quad (11)
$$

and redefine the quadratic design matrix and associated scatter matrices accordingly, then Equation 10 is satisfied by $z_0 = 0$. This effectively forces the hyperplane through the centroid of the data, satisfying the partial derivative with respect to the coordinate z_0. This transformation not only ensures the Euclidean invariance of the fit, but also reduces the dimensionality of the problem. The column of ones, D_0, is redundant to the problem at hand. This is a reduction of dimensionality that has been overlooked in past literature. The problem is reduced to determining the orientation of the hyperplane to be fit, as the relative shift is now known. The reduced system of partial derivatives is now,

$$
\begin{aligned}
\mathsf{S}_{22}\mathbf{z}_2 + \mathsf{S}_{21}\mathbf{z}_1 &= 0 &\quad (12)\\
\mathsf{S}_{21}^T\mathbf{z}_2 + \mathsf{S}_{11}\mathbf{z}_1 &= 0. &\quad (13)
\end{aligned}
$$

Solving the partial derivative with respect to $\mathbf{z}_1$, Equation 13, for the linear coefficient vector yields $\mathbf{z}_1$ when $\mathbf{z}_2$ is held constant, i.e.

$$
\mathbf{z}_1 = -\mathsf{S}_{11}^{-1}\mathsf{S}_{21}^T\mathbf{z}_2. \quad (14)
$$

Substitution of this relation and $z_0 = 0$ into the least squares problem in Equation 6 results in a function in the quadratic coefficients only, and free of the redundant column of ones D_0, i.e.

$$
\mathbf{z}_2^T\left(\mathsf{S}_{22} - \mathsf{S}_{21}\mathsf{S}_{11}^{-1}\mathsf{S}_{21}^T\right)\mathbf{z}_2 = \mathbf{r}^T\mathbf{r}. \quad (15)
$$

The matrix,

$$
\mathsf{M} \triangleq \mathsf{S}_{22} - \mathsf{S}_{21}\mathsf{S}_{11}^{-1}\mathsf{S}_{21}^T = \mathsf{D}_2^T\left(\mathsf{I}_n - \mathsf{D}_1\mathsf{D}_1^+\right)\mathsf{D}_2, \quad (16)
$$

is the reduced scatter matrix sought, and is the *Schur Complement* of S_{11} in the scatter matrix. The matrix D_1^+ is the pseudo-inverse matrix of D_1, i.e. the matrix which maps vector $\mathbf{z}_1$ from a given vector with a least squares residual. The matrix product $\mathsf{D}_1\mathsf{D}_1^+$ is the set of orthogonal projections on to the range space of D_1, and is — to a scaling factor — the covariance of the residuals of the linear portion of the data. Specifically, if σ_1 and σ_2 are the singular values of D_1 which correspond to the residual vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then,

$$\mathsf{D}_1\mathsf{D}_1^+ = \frac{1}{\sigma_1^2}\mathbf{r}_1\mathbf{r}_1^T + \frac{1}{\sigma_2^2}\mathbf{r}_2\mathbf{r}_2^T. \tag{17}$$

This implies that matrix M is the result of subtracting the quadratic residual elements predicted by the linear portion from the residuals of the quadratic portion. This allows an optimization to be performed in the space on the coefficients $\mathbf{z}_2$, a subspace for which corresponding $\mathbf{z}_1$ vectors have a residual of minimal norm. In other words, the mapping in Equation 14 corresponds to $\mathbf{z}_1 = -\mathsf{D}_1^+\mathsf{D}_2\mathbf{z}_2$, and is thus the least squares mapping of $\mathbf{z}_1$ from the residual vector of $\mathbf{z}_2$, and essentially refits the linear portion given the specific quadratic coefficients. This corresponds to the generalization of the Eckart-Young-Mirsky theorem proposed by Golub et al. [5].

4 Fitting Conics with a Quadratic Constraint

The problem of the linear fitting of a conic with a quadratic constraint on the roots at infinity can now be stated as,

$$\mathbf{z}_2^T\mathsf{M}\mathbf{z}_2 = \min_{\mathbf{z}_2\neq\mathbf{0}} \qquad \text{subject to} \qquad \mathbf{z}_2^T\mathsf{C}\mathbf{z}_2 = \alpha, \tag{18}$$

where matrix C is a quadratic constraint on the coefficients a, b, and c. As Bookstein [1] showed, the minimization problem can be stated as a Lagrange multiplier problem, and solved as a generalized eigenvector problem. Combining the function to be minimized and the constraint with a Lagrange multiplier, results in the system

$$H(\mathbf{z}_2) = \mathbf{z}_2^T\mathsf{M}\mathbf{z}_2 + \lambda\left(\mathbf{z}_2^T\mathsf{C}\mathbf{z}_2 - \alpha\right), \tag{19}$$

which is solved for its partial derivatives,

$$\mathsf{M}\mathbf{z}_2 + \lambda\mathsf{C}\mathbf{z}_2 = 0 \tag{20}$$

$$\mathbf{z}_2^T\mathsf{C}\mathbf{z}_2 = \alpha. \tag{21}$$

Solving Equation 20 as a generalized eigenvector problem yields eigenvectors which minimize $\mathbf{z}_2^T\mathsf{M}\mathbf{z}_2$. Moreover, if $(\lambda_i, \mathbf{e}_i)$ is a solution to the generalized eigenvector problem then,

$$\operatorname{sign}(\lambda_i) = \operatorname{sign}\left(\mathbf{e}_i^T\mathsf{C}\mathbf{e}_i\right). \tag{22}$$

This fact is known as the *Sylvester Law of Inertia* [6], and is essential to the specific fitting algorithm. Further simplification occurs if the matrix C is non-singular, in which case the generalized eigenvector problem can be solved as the eigenvector problem,

$$\mathsf{C}^{-1}\mathsf{M}\mathbf{z}_2 = \lambda\mathbf{z}_2. \tag{23}$$

With an approach proposed by O'Leary and Zsombor-Murray [11] the constraint,

$$b^2 - 4ac = \mathbf{z}_2^T\begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}\mathbf{z}_2 = \alpha, \tag{24}$$

is applied. It was shown that two of the resulting eigenvectors correspond to the best elliptical and best hyperbolic solutions. The solutions are extracted by evaluating the condition $b^2 - 4ac$ in terms of the resulting eigenvectors $\mathbf{e}_1$,$\mathbf{e}_2$, and $\mathbf{e}_3$, i.e.

$$\kappa_i = e_{i2}^2 - 4e_{i1}e_{i3}, \tag{25}$$

where e_{ij} is the jth element of the ith eigenvector. The ellipse is found as,

$$_e\mathbf{z}_2 = \mathbf{e}_u \quad \text{where} \quad u = \min_i(\kappa_i), \tag{26}$$

and the hyperbola as,

$$_h\mathbf{z}_2 = \mathbf{e}_v \quad \text{where} \quad v = \min_{i \neq u, \kappa_i > 0} |\lambda_i|, \tag{27}$$

and λ_i is the eigenvalue corresponding to the ith eigenvector. In other words, the ellipse is the eigenvector whose corresponding condition value is the lone negative value. The two positive values correspond to two hyperbolic solutions, whereby the solution with the eigenvalue of minimum magnitude is selected.

5 The Parabola

If one wishes to fit a parabola to scattered data, the eigenvector problem in Equation 23 cannot be applied, as the Lagrange multiplier problem results in the trivial solution when applying the null constraint $b^2 - 4ac = 0$. Also, the secular equation proposed by Gander [4], does not apply when $\alpha = 0$. If the constraint matrix C is the identity matrix, then the system is solved with the constraint $a^2 + b^2 + c^2 = 1$, which is implicit in the evaluation of eigenvector and singular value problems. The resulting eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, with the corresponding eigenvalues λ_1, λ_2, and λ_3, form an orthonormal basis vector set for the space of the quadratic portion of all conics. The eigenvalues and corresponding vectors should be ordered such that,

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq 0, \tag{28}$$

since normally, the eigenvector corresponding to the eigenvalue of smallest magnitude is the best fit solution to the linear conic fit. The above constraint ensures that all solutions lie on the unit sphere centered at the origin. The required condition for a parabola, $b^2 - 4ac = 0$, corresponds to an elliptical cone in this space (see Figure 1) and forms the boundary between the ellipses and hyperbolæ. The curve of intersection of the two quadrics, and thus a fourth order curve, represents the parabolic solutions. Should we want to fit a parabola, we take the quadratic coefficients of the parabola to be a linear combination of the eigenvectors of the matrix M, i.e.

$$_p\mathbf{z}_2 = \mathbf{e}_3 + s\mathbf{e}_2 + t\mathbf{e}_1. \tag{29}$$

Since the singular values of the matrix M, i.e. the square-roots of the eigenvalues of $\mathsf{M}^T\mathsf{M}$, are the 2-norm distances of the respective vectors to the null-space of M, then the eigenvector associated with the smallest singular value can be considered the minimizing solution. Thus, we assume $\mathbf{e}_3$ is the best fit, and use combinations of the other two eigenvectors to find optimal parabolic solutions. Since the equations are homogeneous, only

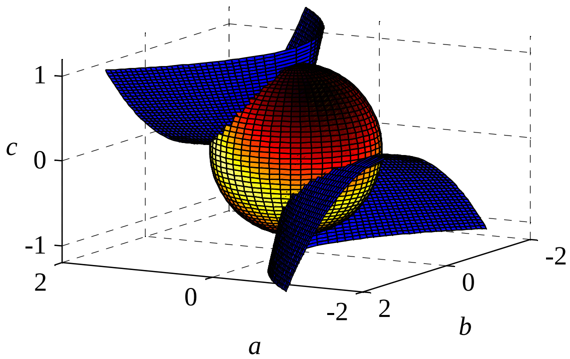

Figure 1: The surface $b^2 - 4ac = 0$ is an elliptical cone which separates the ellipses from the hyperbolæ by the parabolas. The unit sphere representing all solutions shows the error density of a specific set of data on its surface by means of a colour gradient.

two parameters are needed to fully describe the space. The error associated with taking this linear combination is the magnitude of the resulting residual vector, that is,

$$\begin{aligned}\|\mathsf{M}\left({}_p\mathbf{z}_2\right)\| \triangleq \Delta(s,t) &= \left({}_p\mathbf{z}_2\right)^T \mathsf{M}^T \mathsf{M}\left({}_p\mathbf{z}_2\right) \\ &= \left(\mathbf{e}_3 + s\mathbf{e}_2 + t\mathbf{e}_1\right)^T \mathsf{M}^T \mathsf{M}\left(\mathbf{e}_3 + s\mathbf{e}_2 + t\mathbf{e}_1\right) \\ &= \mathbf{e}_3^T \mathsf{M}^T \mathsf{M}\mathbf{e}_3 + \mathbf{e}_2^T \mathsf{M}^T \mathsf{M}\mathbf{e}_2 s^2 + \mathbf{e}_1^T \mathsf{M}^T \mathsf{M}\mathbf{e}_1 t^2 \\ &= \lambda_3^2 + \lambda_2^2 s^2 + \lambda_1^2 t^2. \end{aligned} \tag{30}$$

In the quadratic coefficient space, this error function is essentially an ellipsoid-shaped error density with semi-axes proportional to the singular values of the reduced scatter matrix M. In the space of the parameters s and t, it is an ellipse-shaped error density with semi-axes proportional to the first and second largest singular values. The constraint to ensure that ${}_p\mathbf{z}_2$ is indeed parabolic is found by expanding the constraint $b^2 - 4ac = 0$ in terms of the parametric coefficients, i.e.

$$\left(e_{32} + e_{22}s + e_{12}t\right)^2 - 4\left(e_{31} + e_{21}s + e_{11}t\right)\left(e_{33} + e_{23}s + e_{13}t\right) = 0, \tag{31}$$

where again e_{ij} is the jth element of the ith eigenvector. Expanding this expression yields an expression in the form,

$$C(s,t) = \gamma_1 s^2 + \gamma_2 st + \gamma_3 t^2 + \gamma_4 s + \gamma_5 t + \gamma_6 = 0, \tag{32}$$

which is a conic in the parameter space. The problem is thus to minimize the error function of Equation 30, i.e. $\Delta(s,t)$, upon the points of the constraint conic. This can be formulated as the Lagrange multiplier[1] problem,

$$H(s,t) = \Delta(s,t) + \mu C(s,t). \tag{33}$$

[1]Standard literature uses λ for eigenvalues as well as Lagrange multipliers, thus to avoid confusion, μ has been chosen to denote the Lagrange multiplier.

Upon solving the partial derivatives of $H(s,t)$, a fourth order polynomial in μ is obtained. Defining the coefficients,

$$\begin{array}{lllllllll} \alpha_1 &=& \lambda_1^2 & \alpha_2 &=& \lambda_2^2 & \alpha_3 &=& \alpha_1\alpha_2 \\ k_1 &=& 4\gamma_3\gamma_6-\gamma_5^2 & k_2 &=& \gamma_2\gamma_6-\frac{1}{2}\gamma_4\gamma_5 & k_3 &=& \frac{1}{2}\gamma_2\gamma_5-\gamma_3\gamma_4 \\ k_4 &=& 4\gamma_6\gamma_1-\gamma_4^2 & k_5 &=& 4\gamma_1\gamma_3-\gamma_2^2 & k_6 &=& \gamma_2\gamma_4-2\gamma_1\gamma_5 \\ k_7 &=& -4(\gamma_1\alpha_1+\alpha_2\gamma_3) & k_8 &=& \gamma_1k_1-\gamma_2k_2+\gamma_4k_3, & & & \end{array} \tag{34}$$

the polynomial coefficients are given as,

$$\begin{array}{ll} K_4=k_5k_8 & K_3=2k_7k_8 \\ K_2=4[(2\gamma_2k_2+4k_8)\alpha_3+\gamma_1k_4\alpha_1^2+\gamma_3k_1\alpha_2^2] & \\ K_1=-8\alpha_3(k_1\alpha_2+k_4\alpha_1) & K_0=16\gamma_6\alpha_3^2. \end{array} \tag{35}$$

Thus, solving,

$$K_4\mu^4+K_3\mu^3+K_2\mu^2+K_1\mu+K_0=0, \tag{36}$$

yields four solutions for μ. The best fitting parabola can be extracted as it corresponds usually, but not always, to the real Lagrange multiplier with the smallest magnitude, i.e.

$$\mu_*=\min_i|\mu_i|, \qquad \mu_i\in\mathbb{R}. \tag{37}$$

Backsubstitution for the corresponding s_* and t_* is in the form,

$$s_*=\frac{2\mu_*}{u_*}(k_3\mu_*+\alpha_1\gamma_4), \qquad \text{and} \qquad t_*=\frac{\mu_*}{u_*}(k_6\mu_*+2\alpha_2\gamma_5), \tag{38}$$

where

$$u_*=k_5\mu_*^2+k_7\mu_*+4\alpha_3. \tag{39}$$

The quadratic coefficients of the parabola are found by backsubstitution of the s_* and t_* into the linear combination of the eigenvectors, i.e. ${}_p\mathbf{z}_2=\mathbf{e}_3+s_*\mathbf{e}_2+t_*\mathbf{e}_1$.

6 Backsubstitution

Given the quadratic solution vectors of the conics, $\mathbf{z}_2$, be it the ellipse ${}_e\mathbf{z}_2$, hyperbola ${}_h\mathbf{z}_2$, or parabola ${}_p\mathbf{z}_2$, backsubstitution is the same. The quadratic coefficients are known, and thus the directions of the asymptotes are also known. The backsubstitution then determines the shift of the conic centre as well as its scaling factor as to how far it is from the mere product of the asymptotes, i.e. a degenerate conic. The backsubstitution can be accomplished in concise matrix form, that is,

$$\mathbf{z}=\begin{bmatrix} & \mathsf{I}_3 & \\ & -\mathsf{S}_{11}^{-1}\mathsf{S}_{21}^T & \\ -\overline{x^2} & -\overline{xy} & -\overline{y^2} \end{bmatrix}\mathbf{z}_2 \triangleq \mathsf{B}\mathbf{z}_2, \tag{40}$$

where I_3 is the 3×3 identity matrix. Thus, B is a 6×3 matrix, and the resulting vector $\mathbf{z}$ is the set of corresponding conic coefficients, i.e. the Grassmannian coefficients. As noted above, the mapping of the linear portion is a least squares mapping from the quadratic residual vector, and the constant term corresponds to pushing the hyperplane back to fit through the actual centroid of the data in the hyperspace.

In the plane, the transformation which will place the conic back onto the original data is the same transformation applied to the data but in the form of the similarity transformation $\mathsf{K}_* = \mathsf{T}^T \mathsf{K} \mathsf{T}$, where,

$$\mathsf{T} = \begin{bmatrix} m & 0 & -m\bar{x} \\ 0 & m & -m\bar{y} \\ 0 & 0 & 1 \end{bmatrix}. \tag{41}$$

7 Summary of Algorithm

The algorithm resulting from the above analysis can be summarized as follows:

1. Generate a scaled, mean-free set of data points,

$$_m x_i = m\,(x_i - \bar{x}) \qquad \text{and} \qquad _m y_i = m\,(y_i - \bar{y})\,. \tag{42}$$

2. Perform a linear regression on the mean-free $_m x_i$ and $_m y_i$. If the residual is too small, stop the algorithm, since the data is best described by a line.

3. Generate the quadratic design matrix, and remove the mean values from the columns. Compute the scatter matrix with the linear prediction removed from the mean-free quadratic terms, i.e.

$$\mathsf{M} = \mathsf{S}_{22} - \mathsf{S}_{21}\mathsf{S}_{11}^{-1}\mathsf{S}_{21}^T. \tag{43}$$

4. For ellipses and hyperbolæ, determine the eigenvectors of $\mathsf{C}^{-1}\,\mathsf{M}$, where C defines the constraint $b^2 - 4ac = \alpha$. Select the quadratic portion of the elliptical and hyperbolic solutions by means of the eigenvalues, and values of the constraint evaluated for each eigenvector.

5. For parabolas, solve the unconstrained eigenvector problem. Determine and solve the fourth order polynomial,

$$K_4\mu^4 + K_3\mu^3 + K_2\mu^2 + K_1\mu + K_0 = 0, \tag{44}$$

   and backsubstitute the real μ with the smallest magnitude to obtain the quadratic parabola coefficients.

6. Backsubstitute the quadratic coefficients of the desired conic into $\mathbf{z} = \mathsf{B}\mathbf{z}_2$. Find the conic matrix from the Grassmannian coefficients, and apply the similarity transformation $\mathsf{K}_* = \mathsf{T}^T \mathsf{K} \mathsf{T}$.

8 Numerical tests

The conic forms which are most commonly encountered in metric vision were used to test the algorithm. In each of the five test cases — i.e. elliptical, elliptical arc, hyperbolic, degenerate hyperbolic and parabolic data — all three conic types were fitted. The results show that the algorithm always produces the specific types of conics, regardless of the nature of the data. The tests were performed with random noise with standard deviations of 3% of the amplitude of the respective x and y data. See Figure 2. Further tests indicate that these specific fitting algorithms are more stable than existing, general ones.

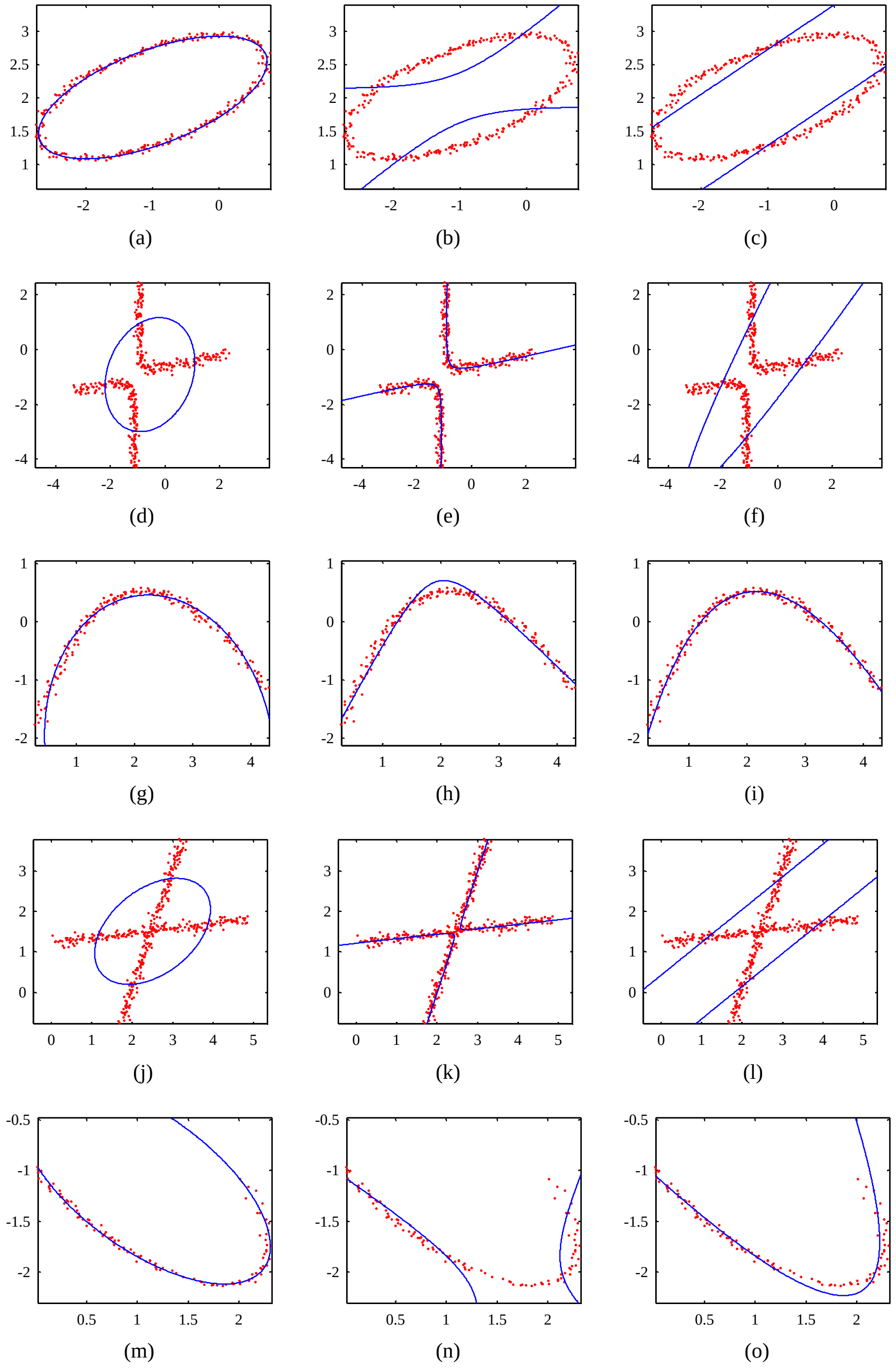

Figure 2: An ellipse (a), hyperbola (b), and parabola (c) fit to noisy elliptical data. An ellipse (d), hyperbola (e), and parabola (f) fit to noisy hyperbolic data. An ellipse (g), hyperbola (h), and parabola (i) fit to noisy parabolic data. An ellipse (j), hyperbola (k), and parabola (l) fit to noisy degenerate data. An ellipse (m), hyperbola (n), and parabola (o) fit to noisy elliptical arc data.

9 Conclusions

The above proposed algorithm provides a new and efficient method for the linear fitting of conics of specific types. The column of ones, common to previous methods is now implicitly in the problem, rather than explicitly. The efficiency arises from this decrease in dimensionality of the problem. The three solutions delivered by the algorithm are also guaranteed to be each the best ellipse, hyperbola, and parabola. The linear and specific fitting has applications in automatic inspection or prejudicial perception, where fast and accurate fitting is required for real time inspection of shape manufacturing.

References

[1] Fred Bookstein. Fitting conic sections to scattered data. *Computer Graphics and Image Processing*, 9:56–71, 1979.

[2] Wojciech Chojnacki, Michael J. Brooks, Anton van den Hengel, and Darren Gawley. Revisiting hartleys normalized eight-point algorithm. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 25(9):1172–1177, 2003.

[3] Andrew Fitzgibbon, Maurizio Pilu, and Robert Fischer. Direct least square fitting of ellipses. *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 21(5):476–480, 1999.

[4] W. Gander. Least squares with a quadratic constraint. *Numer. Math.*, 36:291–307, 1981.

[5] G.H. Golub, A. Hoffman, and G.W. Stewart. A generalization of the eckart-young-mirsky matrix approximation theorem. *Linear Algebra and its Applications*, 88/89:317–327, 1987.

[6] G.H. Golub and C.F. Van Loan. *Matrix Computations*. John Hopkins University Press, Baltimore, third edition, 1996.

[7] Radim Halir and Jan Flusser. Numerically stable direct least squares fitting of ellipses. Technical Report citeseer.nj.nec.com/350661.html, Department of Software Engineering, Charles University, Czech Republic, 2000.

[8] Felix Klein. *Elementary Mathematics from an Advanced Standpoint: Geometry*. Dover Publications, New York, 1940.

[9] Yves Nievergelt. Hyperspheres and hyperplanes fitted seamlessly by algebraic constrained total least-squares. *Linear Algebra and its Applications*, 331:43–59, 2001.

[10] Yves Nievergelt. Fitting conics of specific types to data. *Linear Algebra and its Applications*, 378:1–30, 2004.

[11] Paul O'Leary and Paul Zsombor-Murray. Direct and specific least-square fitting of hyperbolae and ellipses. *Journal of Electronic Imaging*, In press, 2004.