Particular Forms of Homography Matrices

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Abstract

This paper deals with monocular video sequences without calibration to recover a maximum of information on displacement and projection parameters. In this paper, we propose a new way to deal with the huge number of particular cases of homographic relations and validate this approach with some experiments showing that if several models are correct, the model with less parameters gives the best estimation. The experiments presented in this paper show also that even if the motion is approximate, the method is still robust.

1 Introduction

Let us consider uncalibrated monocular video sequences to recover a maximum of information on displacement and projection parameters. This work extends previous studies [22, 13, 14] on particular displacement cases, scene geometry and camera analysis. It focuses on the particular forms of fundamental and homographic matrices.

Several authors have already been interested in particular cases of projection [2, 6, 11, 12, 19, 16], or displacement [10, 5, 21, 3, 20]. Some of them consider several specific cases, compare these different parameterizations, and identify which model corresponds to the provided input data.

The motivations of such studies are threefold: (i) to eliminate singularities of general equations, (ii) to estimate the parameters with more robustness and (iii) to retrieve parameters that cannot be retrieved in the general case.

It is already known that the huge number of particular cases prevent exhaustive studies [13]. Some trial have been done based on tree structures but they are still in development stage. In this paper, we propose a new method to deal with all cases : (i) we use a set of simple rules in order to eliminate some redundant cases and some physical impossible cases, (ii) we divide the set of cases into two sets each corresponding to homographic or fundamental relations and (iii) we divide again the cases into sets corresponding to particular forms. We will provide details for each of these steps in the sections hereafter.

2 Stereo framework

In this section, we review the equations and the formalism of displacement and projection which allow to achieve a minimal parameterization of the relations between 2D points in two frames.

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2.1 Rigid displacements

In this paper, we will consider a rigid or piecewise rigid scene. A 3D-point $\mathbf{M_1} = [X_1\ Y_1\ Z_1\ 1]^T$ is moving onto $\mathbf{M_2} = [X_2\ Y_2\ Z_2\ 1]^T$ by a rotation $\mathbf{R}$ and a translation $\mathbf{t} = [t_0\, t_1\, t_2]^T$: $\mathbf{M_2} = \mathbf{R}\,\mathbf{M_1} + \mathbf{t}$. The $\mathbf{R}$-matrix depends only on three parameters $\mathbf{r} = [r_0\, r_1\, r_2]^T$ related to the rotation angle θ and the rotation axis $\mathbf{u}$ by $\mathbf{r} = 2\ \tan(\frac{\theta}{2})\mathbf{u} \Leftrightarrow \theta = 2\ \arctan(\|\mathbf{r}\|/2)$. The rotation matrix $\mathbf{R} = e^{\mathbf{r}\wedge\cdot} = e^{\tilde{\mathbf{r}}}$ can be developed as a rational Rodrigues formula [17] : $\mathbf{R} = \mathbf{I} + [\tilde{\mathbf{r}} + \frac{1}{2}\tilde{\mathbf{r}}^2]/[1 + \frac{\mathbf{r}^T\cdot\mathbf{r}}{4}]$.

2.2 Camera projection

The most commonly accepted hypothesis states that a 3D-point $\mathbf{M}$ is projected with a perspective projection onto an image plane on a 2D-point $\mathbf{m} = [u\ v\ 1]^T$. Choosing a reference frame attached to the camera, the projection equation is :

$$Z\,\mathbf{m} = \begin{pmatrix} \alpha_u & \gamma & u_0 & 0 \\ 0 & \alpha_v & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathbf{M} \tag{1}$$

where α_u and α_v represent the horizontal and vertical lengths, u_0 and v_0 correspond to the image of the optical center and γ is the skew factor.

This model can be refined, by taking optical distortions into account [18, 4, 7]. In this paper, we will consider that the needed corrections have been done as a preprocessing. Two approximations have been proposed in the literature :

The para-perspective model : The perspective projection model may be approximated [2, 15, 12, 13] to its first order with respect to the 3D coordinates. This is equivalent to approximate the perspective projection in two steps: (i) a projection parallel to the gaze direction onto an auxiliary plane $\mathcal{P}_a$ which is parallel to the image plane and passes through the scene center $\mathbf{M_0} = [X_0\ Y_0\ Z_0]^T$ followed by (ii) a perspective projection onto the image plane. This so called para-perspective model yields linear equations (2).

$$\mathbf{m} = \begin{pmatrix} \alpha_u & \gamma & \beta_u & u_0 \\ 0 & \alpha_v & \beta_v & v_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{M} \tag{2}$$

However, its parameters depend on the gaze direction of the scene (β_u and β_v are related to the other intrinsic parameters and to the gaze direction) :

$$\begin{aligned} \beta_u &= \alpha_u\, X_0/Z_0 + \gamma\, Y_0/Z_0 \\ \beta_v &= \alpha_v\, Y_0/Z_0 \end{aligned} \tag{3}$$

Equation 2 corresponds to the most general case of para-perspective projection although more simple expressions have been proposed [16].

The orthographic model : The zero-order development with respect to the 3D depth consists in a rougher approximation. It is also equivalent to another two steps approximation: (i) an orthogonal projection onto the auxiliary plane $\mathcal{P}_a$ followed by (ii) a perspective projection onto the image plane. This approximation, called the orthographic model (4), is well adapted to foveal attention and is characterized by linear equations without any new parameter. It is an approximation of the para-perspective model when the observed objects are in the fovea, i.e. close to the optical axis :

$$\mathbf{m} = \begin{pmatrix} \alpha_u & \gamma & 0 & u_0 \\ 0 & \alpha_v & 0 & v_0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{M} \tag{4}$$

Those three projection models can be integrated in the following expression :

$$\kappa\ \mathbf{m} = \underbrace{\begin{pmatrix} \alpha_u & \gamma & \lambda\,\beta_u + \mu\,u_0 & (1 \Leftrightarrow \mu)\,u_0 \\ 0 & \alpha_v & \lambda\,\beta_v + \mu\,v_0 & (1 \Leftrightarrow \mu)\,v_0 \\ 0 & 0 & \mu & (1 \Leftrightarrow \mu) \end{pmatrix}}_{\mathbf{A}} \mathbf{M} \tag{5}$$

with :

projection case	λ	μ
perspective projection	1	1
orthographic projection	0	0
para-perspective projection	1	0

2.3 Relations between two frames

Let I_1 and I_2 denote two images. In the general case, there exists a fundamental relation between points $\mathbf{m_2}$ in I_2 and points $\mathbf{m_1}$ in I_1 : $\mathbf{m_2}^T\,\mathbf{F}\,\mathbf{m_1} = \mathbf{0}$ where $\mathbf{F}$ is called the fundamental matrix [9].

However, this relation is not defined in some singular cases. For example, it is well known that, in the perspective projection case, if the displacement is a pure rotation or, if the scene is planar, the relation between points is homographic : $\mathbf{m_2} = \mathbf{H}\,\mathbf{m_1}$ where $\mathbf{H}$ is called the homographic matrix.

2.4 Homographic relation in the para-perspective case

In the para-perspective case, we write the projection and displacement equations by extracting the third column from matrix $\mathbf{A}$:

$$\mathbf{m} = \underbrace{\begin{pmatrix} \alpha_u & \gamma & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{pmatrix}}_{(\mathbf{A})_{-3}} \underbrace{\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}}_{\underline{\mathbf{M}}} + Z \underbrace{\begin{pmatrix} \beta_u \\ \beta_v \\ 1 \end{pmatrix}}_{(\mathbf{A})_3} = (\mathbf{A})_{-3}\,\underline{\mathbf{M}} + Z\,(\mathbf{A})_3$$

where $(\mathbf{A})_{-3}$ is an invertible square matrix since $\det((\mathbf{A})_{-3}) = \alpha_u\,\alpha_v \neq 0$.
Thus $\mathbf{m_1} = (\mathbf{A_1})_{-3}\underline{\mathbf{M}}_\mathbf{1} + Z_1(\mathbf{A_1})_3 \Rightarrow \underline{\mathbf{M}}_\mathbf{1} = ((\mathbf{A_1})_{-3})^{-1}\mathbf{m_1} \Leftrightarrow Z_1((\mathbf{A_1})_{-3})^{-1}(\mathbf{A_1})_3$.
Remember that $\mathbf{m_2} = \mathbf{A_2}\,\mathbf{M_2}$ and $\mathbf{M_2} = [\mathbf{R}|\mathbf{t}]\,\mathbf{M_1}$
Let $\mathbf{K} = (\mathbf{A_2}\,[\mathbf{R}|\mathbf{t}])_3 \Leftrightarrow (\mathbf{A_2}\,[\mathbf{R}|\mathbf{t}])_{-3}\,((\mathbf{A_1})_{-3})^{-1}\,(\mathbf{A_1})_3$
and $\mathbf{H}_{\infty_{para}} = (\mathbf{A_2}\,[\mathbf{R}|\mathbf{t}])_{-3}\,((\mathbf{A_1})_{-3})^{-1}$.
Previous equations lead to : $\mathbf{m_2} = \mathbf{H}_{\infty_{para}}\,\mathbf{m_1} + Z_1\,\mathbf{K}$

This relation is homographic if and only if $\mathbf{K} = 0$ or if there exists a (3×3) matrix $\mathbf{H_Z}$ such as $Z_1\,\mathbf{K} = \mathbf{H_Z}\,\mathbf{m_1}$. The first condition induces a displacement constraint. It leads to the simple equation $\mathbf{r} = \theta\,\mathbf{M_0}$ meaning that the rotation axis is parallel to the gaze direction. The second condition induces a geometric relation on the 3D point : Z_1 is an affine function of X_1 and Y_1, meaning that the 3D points must belong to a plane $\mathcal{P}$, which cannot contain the optical axis and the gaze direction (see [13] for a demonstration).

2.5 Homographic relation in the orthographic case

The orthographic case is a particular case of para-perspective projection for which the gaze direction is the optical axis. Following a demonstration similar to the para-perspective case, we also obtain two constraints; the displacement constraint states that the rotation axis must be parallel to the optical axis, and the geometric constraint states that the 3D-points must belong to the same plane which does not contain the optical axis. All constraints on displacement and scene geometry for homographic relations are summarized in the following table :

projection	**displacement constraint**	**geometric constraint**
perspective	$\mathbf{t} = \mathbf{0}$	plane
para-perspective	$\mathbf{r} \parallel \mathbf{CM_0}$	plane $Z = f(X, Y)$
orthographic	$\mathbf{r} \parallel \mathbf{0z}$	plane $Z = f(X, Y)$

3 Deriving all particular cases

Let us now focus on the exhaustive study of particular cases.

3.1 Particular cases of projection

Let **p1**, **p2** and **p3** denote the different kinds of projection :

p1	$\lambda = 0$ and $\mu = 1$	orthographic
p2	$\lambda = 1$ and $\mu = 0$	para-perspective projection
p3	$\lambda = 1$ and $\mu = 1$	perspective projection

Authors generally make several hypotheses regarding intrinsic parameters. For example, the most general auto-calibration hypothesis states that the intrinsic parameters are constant. They can be known or unknown. Usually, however, some parameters are constant while some others are not.

The **principal point** of coordinates (u_0, v_0) is not fixed at the image plane in the general case but can be fixed in some cases and its position can be known (for example, in the image center). We then change the reference frame, regarding the principal point position.

The $\boldsymbol{\gamma}$ **parameter** is usually assumed to be null or, at least, considered as a constant value. Furthermore, the numerical precision of the model obtained by this parameter is not crucial for the para-perspective or the orthographic projection cases.

Considering the $\boldsymbol{\alpha_u}$ **and** $\boldsymbol{\alpha_v}$ **parameters**, Enciso [8] has experimentally proven that, for a large number of camera, the α_u/α_v ratio can be considered to be a constant value even if other intrinsic parameters change. The constancy of this ratio is expressed by the equality $f = \alpha_u = \alpha_v$ (see [13] for the demonstration).

The $\boldsymbol{\beta_u}$ and $\boldsymbol{\beta_v}$ parameters are null except in the para-perspective projection case. They are related to the other intrinsic parameters by equation 3. Their ratio is : $\beta_u/\beta_v = (\alpha_u X_0 + \gamma Y_0)/(\alpha_v Y_0)$. Thus, if we neglect γ with respect to $\alpha_u X_0/Y_0$, we obtain : $\beta_u/\beta_v = \alpha_u/\alpha_v X_0/Y_0$ which is also a constant ratio, known if X_0/Y_0 value is known.

The following table summarizes, for each intrinsic parameter, the particular cases (constant values are indexed by zero) :

g1	$\gamma = 0$	γ constant and null
g2	$\gamma = \gamma_0$	γ constant
g3	$\gamma = \gamma(\tau)$	γ free
f1	$\alpha_v = 1$	α_v constant and known
f2	$\alpha_v = f_0$	α_v constant
f3	$\alpha_v = \alpha_v(\tau)$	α_v free
s1	$\alpha_u = \alpha_v(\tau)$	$\frac{\alpha_u}{\alpha_v}$ constant and known
s2	$\alpha_u = \alpha_u(\tau)$	α_u free
b1	$\beta_v = 0$	β_v constant and null
b2	$\beta_v = \beta_0$	β_v constant
b3	$\beta_v = \beta_v(\tau)$	β_v free
B1	$\beta_u = \beta_v(\tau)$	β_u and β_v equal
B2	$\beta_u = \beta_v(\tau)$	$\frac{\beta_u}{\beta_v}$ constant
B3	$\beta_u = \beta_u(\tau)$	$\frac{\beta_u}{\beta_v}$ free
c1	$u_0 = v_0 = 0$	u_0 and v_0 constant and known
c2	$u_0 = u_{0_0}$ and $v_0 = v_{0_0}$	u_0 and v_0 constant
c3	$u_0 = u_0(\tau)$ and $v_0 = v_0(\tau)$	u_0 and v_0 free

Table of particular cases of intrinsic parameters for 2 frames

Subsequently, we will refer to each case by the label given in the first column.

3.2 Particular cases of displacement

3.2.1 Discrete motion - continuous motion

In an image sequence, if the displacement between two frames is small, we can approximate the rotation equation by its first order : $\mathbf{R} = e^{\tilde{\mathbf{r}}} = \mathbf{I} + \tilde{\mathbf{r}} + o(\tilde{\mathbf{r}})$, or if the displacement is larger, we can also consider the second order expansion : $\mathbf{R} = \mathbf{I} + \tilde{\mathbf{r}} + \frac{\tilde{\mathbf{r}}^2}{2} + o(\tilde{\mathbf{r}}^2)$

3.2.2 About extrinsic parameters

The rotation parameters are related to the rotation axis and the rotation angle by : $\mathbf{r} = 2\,\tan\frac{\theta}{2}\,\mathbf{u}$ where $\mathbf{u}$ is a unitary vector giving the direction of the rotation axis.

Some components of $\mathbf{u}$ can be known or null. Some values of θ may yield singularities; for example $\theta = 0$ corresponds to a null rotation; $\theta = \frac{\pi}{4}$ and the rotation axis parallel to the translation vector for a screw displacement.

Some robotic systems give precise values of robot displacements (angle, axis, translation); some values may be known, which we indicate with an "_" caracter. Other informations on parallelism or orthogonality to a known direction may be also available. This is also the case for the translation vector. Such relations between axis and direction are considered :

- planar motion: $\mathbf{r} \perp \mathbf{t} \Leftrightarrow \mathbf{r}.\mathbf{t} = 0$
- screw displacement : $\mathbf{r} \parallel \mathbf{t} \Leftrightarrow \exists \kappa \;/\; \mathbf{r} = \kappa \mathbf{t}$
- $\mathbf{r}$ or $\mathbf{t}$ is parallel or orthogonal to a known direction denoted $\_\mathbf{g}$.

3.2.3 All constraints on motion

All these constraints, also called atomic cases, have simple expression that can be easily combined. In this purpose, we use the fact that $\mathbf{u}$ is a unary vector and that, for monocular systems, the norm of translation cannot be recovered. To parameterize these vectors with only 2 parameters, we divide each component by a non-zero component. Then, the dot-product and scalar product induce linear relations. For example, if $t_2 = 1$, $\mathbf{t} \perp \mathbf{r}$ is equivalent to $t_0\, u_0 + t_1\, u_1 + u_2 = 0 \Rightarrow u_2 = \Leftrightarrow t_0\, u_0 \Leftrightarrow t_1\, u_1$

All cases are collected in the following table :

R1	$\mathbf{R} = \mathbf{I}$	null rotation	**W1**	$\mathbf{r}.\_\mathbf{r} = 0$	$\mathbf{r} \perp$ known axis
R2	$\mathbf{R} = \mathbf{I} + \tilde{\mathbf{r}}$	first order	**W2**	$\mathbf{r} \wedge \_\mathbf{r} = 0$	$\mathbf{r} \parallel$ known axis
R3	$\mathbf{R} = \mathbf{I} + \tilde{\mathbf{r}} + \frac{1}{2}\tilde{\mathbf{r}}^2$	second order	**W3**		general case
R4	$\mathbf{R} = \mathbf{I} + \frac{\tilde{\mathbf{r}} + \frac{1}{2}\tilde{\mathbf{r}}^2}{1 + \frac{\mathbf{r}^T\mathbf{r}}{4}}$	general case	**u1**	$u_0 = u_2 = 0, u_1 = 1$	axis $\parallel$ y-axis
r1	$\mathbf{r} = 2\,\tan(\frac{\theta}{2})\,\frac{\mathbf{u}}{\lVert\mathbf{u}\rVert}$	general case	**u2**	$u_0 = 0, u_1 = 1$	axis $\perp$ x-axis
a1	$\theta = \frac{\pi}{2}$	quarter turn	**u3**	$u_2 = 0, u_1 = 1$	axis $\perp$ z-axis
a2	θ	free angle	**u4**	$u_1 = 1$	general case
T1	$\mathbf{t} = \mathbf{0}$	null translation	**u5**	$u_0 = u_2 = 0, u_1 = -1$	axis $\parallel$ y-axis
T2	$\mathbf{t} = [t_0\ t_1\ t_2]^T$	translation	**u6**	$u_0 = 0, u_1 = -1$	axis $\perp$ x-axis
t1	$t_1 = t_2 = 0, t_0 = 1$	trans. $\parallel$ x-axis	**u7**	$u_2 = 0, u_1 = -1$	axis $\perp$ z-axis
t2	$t_1 = 0, t_0 = 1$	trans. $\perp$ y-axis	**u8**	$u_1 = -1$	general case
t3	$t_2 = 0, t_0 = 1$	trans. $\perp$ z-axis	**u9**	$u_0 = u_1 = 0, u_2 = 1$	axis $\parallel$ z-axis
t4	$t_0 = 1$	general trans.	**u10**	$u_0 = 0, u_2 = 1$	axis $\perp$ x-axis
t5	$t_0 = t_2 = 0, t_1 = 1$	trans. $\parallel$ y-axis	**u11**	$u_1 = 0, u_2 = 1$	axis $\perp$ y-axis
t6	$t_0 = 0, t_1 = 1$	trans. $\perp$ x-axis	**u12**	$u_2 = 1$	general case
t7	$t_2 = 0, t_1 = 1$	trans. $\perp$ z-axis	**u13**	$u_0 = u_1 = 0, u_2 = -1$	axis $\parallel$ z-axis
t8	$t_1 = 1$	general trans.	**u14**	$u_0 = 0, u_2 = -1$	axis $\perp$ x-axis
t9	$t_0 = t_1 = 0, t_2 = 1$	trans. $\parallel$ z-axis	**u15**	$u_1 = 0, u_2 = -1$	axis $\perp$ y-axis
t10	$t_0 = 0, t_2 = 1$	trans. $\perp$ x-axis	**u16**	$u_2 = -1$	general case
t11	$t_1 = 0, t_2 = 1$	trans. $\perp$ y-axis	**u17**	$u_1 = u_2 = 0, u_0 = 1$	axis $\parallel$ x-axis
t12	$t_2 = 1$	general trans.	**u18**	$u_1 = 0, u_0 = 1$	axis $\perp$ y-axis
D1	$\mathbf{t}.\_\mathbf{t} = 0$	$\mathbf{t} \perp$ known axis	**u19**	$u_2 = 0, u_0 = 1$	axis $\perp$ z-axis
D2	$\mathbf{t} \wedge \_\mathbf{t} = 0$	$\mathbf{t}.$ $\parallel$ known axis	**u20**	$u_0 = 1$	general case
D3		no relation	**u21**	$u_1 = u_2 = 0, u_0 = -1$	axis $\parallel$ x-axis
Z1	$\mathbf{t}.\mathbf{u} = 0$	$\mathbf{t} \perp$ rotat. axis	**u22**	$u_1 = 0, u_0 = -1$	axis $\perp$ y-axis
Z2	$\mathbf{t} \wedge \mathbf{u} = 0$	screw displ.	**u23**	$u_2 = 0, u_0 = -1$	axis $\perp$ z-axis
Z3		no relation	**u24**	$u_0 = -1$	general case

Table of particular cases of displacements

3.3 Generating all cases

All particular cases, each called a molecular case, are generated by combining[1] the atomic cases and solving the constraints by substitution with some rules: one projection mode, one rotation mode. . . This corresponds to choose one case in each family, a family being named by a letter. Thus, a molecular case is identified by the sequence :

[1]This work is done using Maple for symbolic computation.

```
p[1-3]g[1-3]f[1-3]s[1-3]b[1-3]B[1-3]c[1-3]R[1-4]r1a[1-2]u[1-24]W[1-3]T[1-2]
t[1-12]D[1-3]Z[1-3]
```

How many cases do we have? If we look at the expression of a particular case above-mentioned, we obtain 3.10^8 particular cases. However, this is not the real number of particular cases because of **incompability** and **redundancy** of some combinations of constraints.

It is easy to eliminate incompatible constraints but it is not possible to deal with redundant constraints because it requires to compare each set of combined constraint with the others to determine the similarity. The complexity of this process is $O(n^2)$.

Although we cannot remove redundant cases, we propose an adapted strategy to deal with the number of cases. Previous works have tried to build a hierarchy but they encounter problems to manage it. The idea of this paper is (i) to eliminate some of the redundant cases by some considerations on the atomic cases and (ii) to limit the number of cases by the study of the particular forms of the matrices. For this second step, we will separate cases into two subgroups: cases inducing homographies and cases inducing fundamental relations.

3.4 Reducing the number of cases

Some redundancy are obvious :

- in case (`R1`), one case of axis and angle is condidered,
- in cases (`R2`) and (`R3`), we do not consider (`a1`) when θ is equal to $\frac{\pi}{2}$,
- the case (`a1`) is only considered if $\mathbf{r} \parallel \mathbf{t}$, (`Z2`),
- in case $\mathbf{t} = \mathbf{0}$, we do not consider any relation of orthogonality or parallelism,
- in cases (`p1`) and (`p3`), β_u and β_v are equal to zero.

We also consider the following experimental simplifications :

- in cases (`p1`) and (`p2`), we neglect γ with respect to other approximations,
- we assume that the ratio α_u/α_v is constant,
- these two previous items imply that β_u/β_v is also constant.

Then, it remains 2539953 particular cases. This is approximately 100 times less than previously determined.

3.5 Fundamental and homographic matrices

As previously studied in subsections 2.3, 2.4 and 2.5, the displacements inducing homographic relations are :

- in the orthographic case (`p1`) : $\mathbf{u} \parallel Oz$. The relations between $\mathbf{t}$ and $\mathbf{r}$ are equivalent to the nullity of some vector components. We will not consider (`Z1`) and (`Z2`). Previous studies on orthographic displacement have shown that the displacement is retinal (`t[1;3;5;7]`).
- in the para-perspective case (`p2`) : $\mathbf{u} \parallel [X_0\ Y_0\ Z_0]$ (`D2`). Since the view axis has at least a component on the optical axis, we set that $u_2 = \pm 1$. Since the view axis is not exactly the optical axis, we cannot have $u_0 = 0$ and $u_1 = 0$.

- in the perspective case (p3) : $\mathbf{t} = \mathbf{0}$. We thus do not consider the parallelism and orthogonality constraints on $\mathbf{t}$.

We also note that, since we are dealing with only 2 views, relations between $\mathbf{r}$ or $\mathbf{t}$ with a known vector $\vec{g}$ will not simplify the $\mathbf{H}$-matrix form, except in the para-perspective case, if $\vec{g} = \mathbf{M_0}$.

For homographic relations, it leads a total of 21330 cases. For fundamental matrices, we will not study para-perspective and orthographic projections since the domain of validity of such projection approximations is included in conditions of existence of homographic relation. In the case of perspective projection, we obtain 72252 cases.

4 Forms of homography matrices

We have significantly reduced the number of cases. We split homographic relations in sets of matrices by forms. We determine a matrix form by a very simple parameterization. We consider (3×3) matrices having 9 parameters (coefficients). If a coefficient is equal to zero, then, there is one less parameter. If a coefficient has the same expression or is opposite to another, there is one less parameter again. These operations are very simple and can be rapidly computed on every cases. Obviously, we know that an homographic matrix is defined up to a scale factor but we will eliminate this parameter at the numerical stage only. This process reduces the 21330 cases to only 108 subgroups. We have established a **table of reduced forms**[2] showing the simplified forms obtained and, for each form, all cases that have generated them.

5 Experiments

We have recorded several video sequences for which the camera displacement induces an homographic relation between image points $\mathbf{m_1}$ and $\mathbf{m_2}$. From each matrix form enumerated in table of reduced forms, we have estimated the homography parameters with the robust least median square method to minimize the distance between a 2D point $\mathbf{m_1}$ and his projected estimation $\mathbf{H}\,\mathbf{m_2}$. To deal with cases with different degrees of freedom, we use an appropriate Akaike criterion [1].

For each video sequence, we have verified that the model with the less residual error effectively corresponds to the displacement performed by a robotic system. An example is proposed in figure 1. For each two consecutive images, the case with less residual error is the case n^{o}51 in table of reduced forms that corresponds to the matrix form :

$$\mathbf{H}_{51} = \begin{pmatrix} x_1 & x_2 & x_3 \\ -x_2 & x_1 & x_6 \\ 0 & 0 & x_1 \end{pmatrix}$$

We observe that this case corresponds to a first order rotation (R2). If we consider only the first and the last frame, the rotation is general (R4).

After that, we performed several experiments without any precise robotic system. A human took a camera by hand and tried to do several particular displacement. We show in

[2] `http://www.inria-sop.fr/robotvis/personnel/dlingran/bmvc-table.ps.gz`

Figure 1: Frames 1, 2, and 8 of the video sequence. The robotic system performs a rotation around the optical axis.

Figure 2: Approximate rotation around the optical axis and translation.

figure 2 two frames of a video sequence. The camera performed approximately a rotation around its optical axis and a translation. As the previous experiment with a robotic system, for each two consecutive images, the case with less residual error is the case n^{o}51 in table of reduced forms. This result shows the robustness of the analysis of displacement by particular cases.

6 Conclusion

We have determined the conditions of existence of homographic relations between projected 2D points for the orthographic, the para-perspective and the perspective projections. We have used these conditions and other obvious redundancy properties to reduce the amount of homographic particular cases to study. Thus, we have determined all particular forms of matrices and we have obtained, for each particular form, the list of cases that have generated this form. This result is a first fundamental step for further studies.

This study might be completed in two ways : (i) to be able, given a matrix form, to analyze the molecular constraints, to determine which are redundant and which correspond to the case we are dealing with, and, (ii) to do the same analysis with geometrical property of the 3D scene, meaning homography induced by planes. The structure of this analysis is as general as possible in order to extend this work to other kind of cameras.

The applications are twofold: an incremental reconstruction of scene and segmentation of objects performing different displacements or with different geometric properties in video sequences.

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