Affine Integral Invariants for Extracting Symmetry Axes

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Abstract

In this paper, we propose integral invariants based on group invariant parameterisation. The new invariants do not suffer from occlusion problems, do not require any correspondence of image features unlike existing algebraic invariants, and are less sensitive to noise than differential invariants. Our framework applies affine differential geometry to derive novel affine integral invariants. The new invariants are exploited for extracting the symmetry axes of planar objects viewed under weak perspective. The proposed method is tested on natural leaves and is shown to extract symmetry axes reliably.

1 Introduction

Recent progress in computer vision has revealed the importance of invariants in object recognition and identification [9, 12, 13]. Existing invariants applied in computer vision are of two types: algebraic [9, 14] and differential invariants [3, 13]. A good example of algebraic invariants is the cross ratio, the ratio of ratios of length, and is invariant under projective transformations. On the other hand, the group curvatures (e.g. Euclidean and affine curvature) and their derivatives are invariant under group transformations, and are called differential invariants.

It is also useful to classify invariants from their area of definition. The globally defined invariants such as the moment invariants [6, 11] are called global invariants, while the differential invariants are local invariants. Either type of invariant has advantages and disadvantages in computer vision applications. To motivate a new framework of invariants we now summarise the properties of these existing invariants based on important requirements in computer vision applications.

1. **Noise Sensitivity:**
   The differential invariants often require high order derivatives. For example, affine curvature requires fourth-order derivatives [5] and the projective differential invariants require eighth-order derivatives [13]. Although, high order derivatives can be computed by fitting polynomial curves to the image features [13], the computed derivatives are unstable and small noise in the image causes large error in invariants. The algebraic invariants, on the other hand, do not require any derivatives and are thus much less sensitive to noise than the differential invariants.
2. **Correspondence:**
   The advantage of the differential invariants is that these are defined locally, and thus do not require any correspondence of distinguished image features (corner points, inflection points, bi-tangents) to be established, while the algebraic invariants or semi-differential invariants require exhaustive search of the corresponding groups of points or lines. The continuity and the proximity of the image features are often used for finding groups efficiently and thus reducing the

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number of iterations required [9]. These heuristics however sometimes derive completely wrong groups and make the systems unstable. The semi-differential invariants have recently been proposed [12] to reduce the number of the corresponding points required without using high order derivatives. Although the inflection points or the bi-tangents of curves can be used as an index for the semi-differential invariants, these points are not always available in images, and furthermore, finding the correspondence of these points still remains a problem.

3. **Occlusion:**
   The locality of invariants is quite important for the occlusion problem. Local invariants are defined at a single point on curves, and thus can be computed even though the remaining parts of the curves are occluded or disappear partially, while global invariants, especially moment invariants [6, 11], severely suffer from occlusion.
4. **Distinguishability:**
   Locality and globality of invariants are also important for distinguishability of the invariants. If we look at a curve locally, it is sometimes quite hard to distinguish a corresponding point from the others. This is because local structures often do not have enough information to distinguish points on curves. This is closely related to the importance of *geometric saliency* for curve matching proposed by Cham and Cipolla [2].

In this paper, we propose integral invariants based on group invariant parameterisation. Unlike the traditional integral invariants (i.e. moment invariants), the new integral invariants are defined semi-locally and therefore do not suffer from the occlusion problem. Unlike the algebraic invariants, the new integral invariants do not require any correspondence of image features. Furthermore, since the area of integration is controllable, we can represent the curves by invariant signatures in many different scales, and can therefore choose the best scale raising the distinguishability. We apply the affine differential geometry to this framework of integral invariants and derive useful invariants under equi-affine or special affine transformations (i.e. affine transformations whose determinant is equal to one). The proposed invariants can successfully be applied for extracting the symmetry axes of planar objects.

2 Semi-Local Integral Invariants

In this section, we investigate a new framework of integral invariants under group transformations, which, unlike the algebraic invariants, do not require any correspondence of the distinguished image features and are less sensitive to noise than the classical differential invariants.

2.1 Condition of Invariance in Integral Formula

Consider a curve, C, to be parameterised by w. Let I be a line integral of a function, F, along C with interval of $[w_1, w_2]$:

$$I = \int_{w_1}^{w_2} F dw \tag{1}$$

We first state a condition of invariance of an arbitrary function I which is well known in the Lie Group theorem. Let $\mathbf{v}$ be an infinitesimal generator of the group transformation. The following theorem holds [8]:

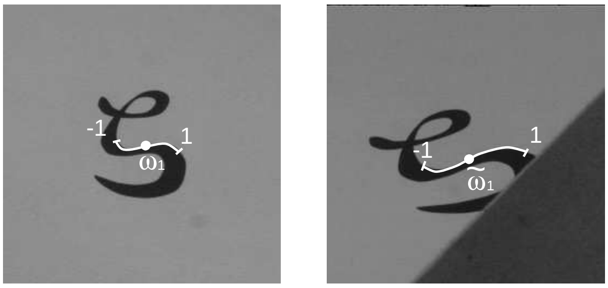

Figure 1: Interval of integration can be identified uniquely from invariant arc-length, w. For example, if w_1 and $\widetilde{w}_1$ are corresponding each other, the interval [-1, 1] with respect to w is corresponding to the interval [-1, 1] with respect to $\widetilde{w}$ in the second image. Even though the curve is occluded partially (second image), the semi-local integral invariants can be defined on the remaining parts of the curve.

Theorem 1 *Let G be a group of transformations. A real-valued function I is invariant under group transformations, if and only if the Lie derivative of I with respect to any infinitesimal generator, $\mathbf{v}$, of the group, G, vanishs as follows:*

$$\pounds_{\mathbf{v}}(I) = 0 \tag{2}$$

where $\pounds_{\mathbf{v}}(\cdot)$ denotes the Lie derivative with respect to a vector field $\mathbf{v}$.

The important property of this theorem is that it guarantees invariance not only under infinitesimal transformations but also under general transformations of the group in spite of its description in infinitesimal criterion.

Substituting (1) into (2), we have the following condition of invariance in integral formula:

Proposition 1 *The integral of a function, F, with respect to a parameter, w, is invariant under group transformations, if and only if the following identity holds for any infinitesimal generator, $\mathbf{v}$, of the group:*

$$\pounds_{\mathbf{v}}(F)dw + F\pounds_{\mathbf{v}}(dw) = 0 \tag{3}$$

In practice, since dw includes derivatives, we must compute the Lie derivatives with respect to the *prolonged* vector fields [8] which transform the derivatives of dw to the derivatives of the transformed dw. We assume in this paper that the vector fields are prolonged as required.

2.2 Semi-Local Integral Invariants

The above proposition states the general condition that must be satisfied for integral invariants under group transformations. The special case of the above proposition can be stated as follows.

Proposition 2 *The integral of a function, F, with respect to a parameter, w, is invariant under group transformations, if the following two equations hold for any infinitesimal generator, $\mathbf{v}$, of the group:*

$$\pounds_{\mathbf{v}}(F) = 0, \qquad \pounds_{\mathbf{v}}(dw) = 0 \tag{4}$$

Since the Lie derivatives commute differentials, the above statement means that if both F and w are invariant under the group transformation, the integral, I, in (1) is also invariant under the group transformation.

The simplest integral invariants can be derived by substituting $F = 1$ into (1):

$$w = \int_{w_1}^{w_2} dw$$

which is the so called group arc-length between $C(w_1)$ and $C(w_2)$, and is invariant under group transformations as follows:

$$\widetilde{w} = w \tag{5}$$

where, the symbol, $\widetilde{\ }$, denotes components transformed by group transformations. Consider a point, $C(w_1)$, on a curve C to be transformed to a point, $\widetilde{C}(\widetilde{w}_1)$, on a curve $\widetilde{C}$ by a group transformation as shown in Fig. 1. From (5), it is clear that if we take the same interval $[-\Delta w, \Delta w]$ around $C(w_1)$ and $\widetilde{C}(\widetilde{w}_1)$, then these two intervals are corresponding each other (see Fig. 1). That is, integrating with respect to the group arc-length, the corresponding interval of integration of the original and the transformed curves can be identified uniquely without using any heuristics. This is a very important property in computer vision applications, since it enables us to define integral invariants even though the curves are partially occluded. For example, in Fig. 1, the curve is occluded in the transformed image. The traditional moment invariants do not work, since in this case the finite support assumption is broken, while proposed integral invariants can be defined semi-locally on the remaining parts (visible parts) of the curve, and can be used for matching the visible parts of the curves.

We now define semi-local integral invariants at point $C(w_1)$ with interval $[-\Delta w, \Delta w]$ as follows:

$$I(w_1) = \int_{w_1-\Delta w}^{w_1+\Delta w} F dw \tag{6}$$

Note we can take an arbitrary interval, Δw, in this formula, that is theoretically we can derive infinite number of independent invariants by just taking different interval, Δw in this equation. This property also enables us to choose the appropriate scale of observation.

3 Affine Differential Geometry

Up to now we have investigated group invariant integral which do not suffer from occlusion problems. We now apply affine differential geometry to this framework and derive integral invariants under affine transformations. We first review well known results in affine differential geometry [5].

Consider a smooth planar curve, $C \in \mathbf{R}^2$, parameterised by p to be transformed to $\widetilde{C} \in \mathbf{R}^2$ by a 2D affine transformation, $A \in GL(2)$, as follows:

$$\widetilde{C} = AC$$

where, the symbol, $\widetilde{\ }$, denotes the components transformed by an affine transformation. The curve is also parameterised by a non-trivial parameter, s, which does not change under a special affine transformation. This parameter is called affine arc-length, and satisfies [5]:

$$[C_s, C_{ss}] = 1 \tag{7}$$

where, C_s and C_{ss} denote the first and second derivatives of C with respect to s, and $[\mathbf{v}_1, \mathbf{v}_2]$ denotes the determinant of a matrix which consists of two column vectors, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^2$. Applying the chain rule to (7), the differential, ds, of affine arc-length can be described as follows:

$$ds = [C_p, C_{pp}]^{\frac{1}{3}} dp \tag{8}$$

where, C_p and C_{pp} denote the first and second derivatives of C with respect to p. Differentiating (7), and after some manipulation, we arrive at the definition of affine curvature, μ:

$$\mu \quad = \quad [C_{ss}, C_{sss}] \tag{9}$$

It is easy to show that both ds and μ are absolute invariants under special affine transformations and relative invariants under general affine transformations as follows:

$$d\tilde{s} = [A]^{\frac{1}{3}} ds \qquad \widetilde{\mu} = [A]^{-\frac{2}{3}} \mu. \tag{10}$$

where, $[A]$ denotes determinant of A. ds and μ are the first and the second lowest order affine differential invariants.

As we have seen in (8) and (9), special affine differential invariants require fourth-order derivatives with respect to p. Higher order derivatives are required for more than two independent invariants. In the next section, we apply affine differential geometry to the framework of integral invariants and derive affine integral invariants which can be computed from lower order derivatives (up to second only) than affine differential invariants. These novel invariants will preserve the advantage of local invariants, that is correspondence free and the tolerance to the occlusion.

4 Affine Semi-Local Integral Invariants

In this section, we derive integral invariants under special affine transformations, which unlike the algebraic invariants do not require any correspondence of distinguished image features and are less sensitive to noise than pure differential invariants. For general affine case, see [10].

As we have seen in (10), the affine arc-length, s, is an absolute invariant under special affine transformations ($[A] = 1$). We can therefore use affine arc-length as the invariant parameter for the semi-local integral invariants. By substituting s for w in (6), we have the following integral invariants under special affine transformations:

$$I(s_1) = \int_{s_1 - \Delta s}^{s_1 + \Delta s} F ds \tag{11}$$

What function F should we choose in this formula? The answer is that any invariant under special affine transformations can be applied. Some examples of F and the derived invariants are now given:

1. **Using Affine Curvature:**
   The simplest integral invariant is derived by substituting $F(s) = \mu(s)$ into (11):

   $$I_1(s_1) = \int_{s_1 - \Delta s}^{s_1 + \Delta s} \mu(s) ds$$

   which is an absolute invariant under special affine transformations. Although this invariant is less sensitive to noise than the affine curvature itself, the order of the derivatives required is still high.

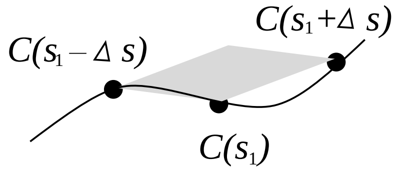

Figure 2: The integral invariant $I_3(s_1)$ is same as the area made by two vectors, $C(s_1 + \Delta s) - C(s_1)$ and and $C(s_1 - \Delta s) - C(s_1)$ (hatched area).

2. **Using Semi-Differential Invariants:**
   Up to now we integrated pure differential invariants. It is however also possible to combine algebraic and differential invariants (i.e. semi-differential invariants) in this framework. The derivative of C at s_1 and the relative position between $C(s_1)$ and a moving point, $C(s)$, constitute a semi-differential invariant [12], $F(s) = [C(s) - C(s_1), C_s(s_1)]$, where, $C_s(s_1)$ denotes the first derivative of C with respect to s at s_1, and s changes from $s_1 - \Delta s$ to $s_1 + \Delta s$. Substituting $F(s)$ into (11), we have:

$$I_2(s_1) \quad = \quad \int_{s_1 - \Delta s}^{s_1 + \Delta s} [C(s) - C(s_1), C_s(s_1)] ds$$

   which is an absolute invariant under special affine transformations. A similar results have been reported by Van Gool et al. [12]. The major difference is that they used inflection points as reference points, while we have used moving points which relative position can be identified uniquely by using affine arc-length. Van Gool's method requires heuristic search of corresponding points, while our method does not require any search.
3. **Analytically Solvable Cases:**
   If we choose the function F carefully, the integral formula (11) can be solved analytically, and the resulting invariants have simpler forms. For example, if we substitute $F(s) = [C_s(s), C(s_1 + \Delta s) - C(s_1)]$ into (11), the integral formula is solved analytically, and the invariant can be described by:

$$I_3(s_1) = [C(s_1 + \Delta s) - C(s_1), C(s_1 - \Delta s) - C(s_1)] \qquad (12)$$

   which is actually the area made by two vectors, $C(s_1 + \Delta s) - C(s_1)$ and $C(s_1 - \Delta s) - C(s_1)$, as shown in Fig. 2. Similar results have been proposed by Bruckstein [1] by a different approach. The invariant requires only the integral of the second order derivatives, and thus its noise sensitivity is similar to that of the first order derivatives, while the noise sensitivity of the pure differential invariants (i.e. affine curvature) requires fourth order derivatives.

Procedure for matching two curve segments

1. B-spline curves are fitted to Canny edge data of each curve.
2. The affine arc-length and the affine integral invariant (12) with an arbitrary but constant Δs are computed at all points on both curves, and plotted on an invariant graph with the horizontal axis of the affine arc-length and the vertical axis of the integral invariant. The derived curves on the graph are invariant signatures up to a horizontal shift.

3. To match curves we simply shift one invariant signature horizontally minimising the total difference between two signatures.
4. Corresponding points are derived by taking identical points on these two signatures.
5. As we will describe in the next section, the second invariant signature is derived by just reflecting the original signature for extracting the symmetry axes of a curve.

5 Experiments

In this section, we apply the proposed integral invariants for extracting the symmetry axes of planer objects. It is known that the bilateral symmetry under weak perspective (skewed symmetric curves) can be described by special affine (equi-affine) transformations with determinant of -1 [7]. Thus if we reflect an invariant signature of a symmetric curve with respect to the vertical axis and put it together with the original invariant signature, then these two signatures are identical. Even though the curve is partially occluded or partially asymmetric, the corresponding symmetric points on the curve can be extracted automatically by just taking the identical points in the original and the reflected signatures.

Fig. 3 (a) shows the image of a natural leaf with symmetric contours. The contour curve extracted by B-spline fitting [2] is shown by a thin line. Since the leaf is nearly flat and the extent of the leaf is much less than the distance from the camera to the leaf, we can assume the corresponding symmetric parts of the contour are related by a special affine transformation. The solid lines in Fig. 3 (b), (c) and (d) show the invariant signatures of the contour computed from the proposed method in three different scales (interval of integration). As shown in these signatures, if the scale is too small, the distinguishability of the invariant signature degrades, and if the scale is too large, asymmetric parts which are often caused by occlusion induce error in the signatures. The dashed lines in Fig. 3 (b), (c) and (d) show the signatures derived by reflecting the original signatures with respect to the vertical axes. The corresponding symmetric points on the contour curve can be extracted by just taking the identical points in these two signatures (solid and dashed lines). The correspondences of symmetric points extracted from Fig. 3 (c) are shown in Fig. 3 (e) by thin lines. Since a point which bisects corresponding symmetric points lies on the symmetry axis, we can derive the symmetry axis by computing the best fit line to the points which bisect corresponding symmetric points. The thick line in Fig. 3 (e) shows the extracted symmetry axis of the leaf. Note that even though the extracted contour (thin line in Fig. 3 (a)) includes asymmetric parts, fairly accurate symmetry axis has been extracted. This is because symmetric parts on the curves can be distinguished from asymmetric parts by using invariant signatures. In Fig. 4, we show the results from another symmetric object. Even though the object is partially out of the scene, the extracted symmetry axis is accurate, while pure global methods, i.e. moment method [4], do not work in such cases. Fig. 5 compares the noise sensitivity of the proposed invariants with that of differential invariants. We find that the proposed invariants are much less sensitive to noise. This is because the proposed invariants require only low order derivatives (2nd order), while the differential invariants require higher order derivatives (4th order). These results confirm the power of the proposed method.

6 Discussion

We now summarise and discuss the properties of the proposed framework of invariants.

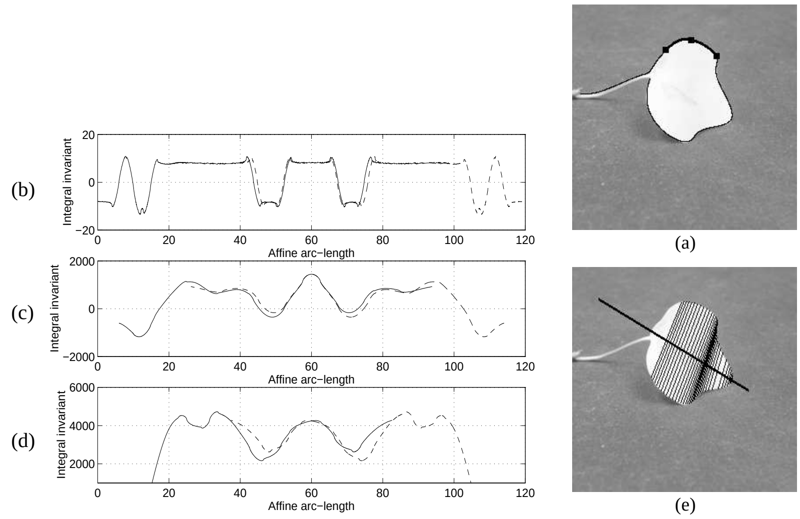

Figure 3: The image of a natural leaf (a). The thin black line shows contour curves extracted by B-spline fitting [2] to Canny edge data. Invariant signatures of the curve derived from three different interval of integration are shown in (b), (c) and (d). The interval of integration is (b) 4.0, (c) 20.0 and (d) 40.0 (the interval of 20.0 is shown by a thick line in (a)). The solid lines in these graphs show original signatures and the dashed lines show the signatures reflected with respect to the vertical axes. If the interval is too small (b), the signature is monotonic, while if the interval is too large (d), asymmetric parts cause error in signature of symmetric parts. The thin lines in (e) show the corresponding symmetric points on the curve extracted by the invariant signatures (c). The symmetry axis extracted from the proposed method is shown in (e) by a thick line.

1. **Noise Sensitivity:**
   The proposed invariants require only the integral of the second derivatives, and the noise sensitivity is therefore close to that of the first derivatives, while the pure differential invariants require fourth derivatives for special affine invariants. The noise sensitivity of the proposed method is thus much better than that of the differential invariants.
2. **Correspondence:**
   Since the invariant parameterisation guarantees unique identification of corresponding interval for integration, the proposed method does not require any heuristic search of corresponding reference points for computing invariants. This is a big advantage especially in complex scenes.
3. **Occlusion:**
   The traditional integral invariants do not work under partial occlusion unless the occluded area is identified. In this research, we have shown that it is actually possible to define integral invariants under partial occlusion without using any heuristics by using invariant parameterisation. We believe this property is quite important especially for curve matching in images under relative motion between the observer and the scene, where we have a lot of unpredictable occlusions.

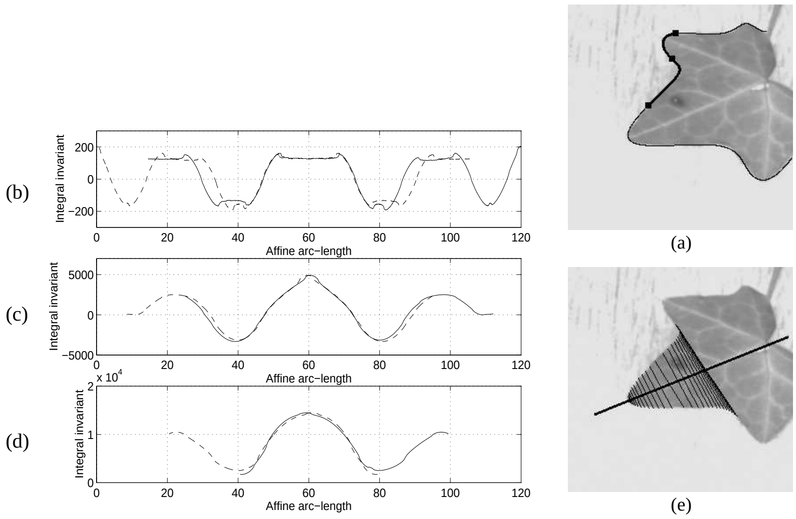

Figure 4: Another example of a natural leaf which is partially out of the scene. See Fig. 3 for the caption.

4. **Distinguishability:**
Another good property of the proposed invariants is that the scale of observation (i.e. the interval for integration) is controllable. This can be done by changing Δw in (6). If the scale of observation is too small (i.e. too local), the invariants do not have enough distinguishability, whereas if it is too large (i.e. too global), the invariants suffer from the occlusion problem as we have seen in Fig. 3. The extreme case of former is the differential invariants, and the latter is the classical moment invariants. The proposed method provides us invariants which are somewhere between these two and have both enough distinguishability and tolerance to the occlusion. The choice of scale remains to be investigated.

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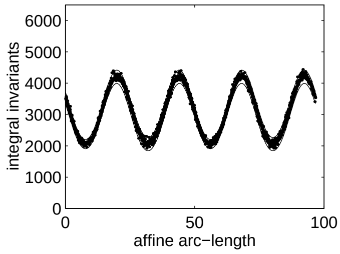
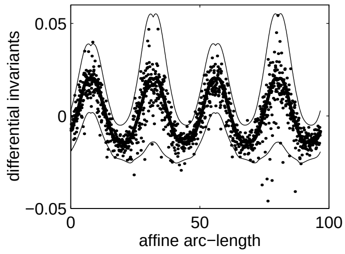


(a) proposed invariants



(b) differential invariants

Figure 5: The invariant signatures of an artificial curve derived from the proposed invariants (semi-local integral invariants) and the differential invariants (affine curvature) are shown by thick lines in (a) and (b) respectively. The dots show signatures after adding random noise of std 0.5 pixels to the curve data. The thin lines show the results of noise sensitivity analysis (3σ) computed from the perturbation method. We find that the proposed invariants are much less sensitive to noise.

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