Interpreting trihedral vertices by using assumptions about the angles between the edges

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This paper considers the geometry of mapping a trihedral vertex onto lines in an image. Where previous analyses of the issue have concentrated on the problem of fitting fully specified models, ours is motivated by considering the assumptions that human vision uses to interpret images of corners, particularly assumptions about the angles between edges. We describe evidence that assumptions of that kind are used in human vision. We then develop an analysis which makes explicit the relationship between projected angles and known or assumed angles at the vertex. This can be used directly to arrive at interpretations. It also allows us to identify limiting cases which show how a projection can be used to constrain possible hypotheses about the angles at vertices.

Trihedral corners are common enough to be worth understanding properly. This paper considers the geometry of mapping a trihedral vertex onto lines in an image. Some of the basic equations which describe that transformation are already known [1], but they have been incompletely explored. This incompleteness is related to the way the equations have been used, which involves mapping a fully specified model onto the picture.

This paper is motivated by indications that human vision uses less restrictive assumptions about corners to achieve interpretation, in the context of changing images as well as static ones. A type of assumption which seems particularly relevant is that trihedral corners involve angles with special attributes. We explore the geometry of trihedral corners with a view to using that kind of assumption. The paper deals with parallel projection: perspective introduces additional issues, but they can be handled by straightforward extensions.

COMBINATIONS OF ANGLES WHICH ARE FAVOURED IN HUMAN VISION

The most natural type of assumption about the angles at a vertex is that all three angles are right. The key relationships in that case were demonstrated long ago. Perkins [2] identified qualitative relationships which determine whether three picture lines can represent a vertex containing three right angles - which he called a cubic corner. If a junction represents a cubic corner, then either two of its angles are acute and sum to more than 90°; or else all three of its angles are obtuse. Attneave and Frost [3] described equations which derive the slopes of a vertex's edges in 3-d space on the assumption that it represents a cubic corner. If s is the slope of an edge in 3-space and **A** and **B** are the angles flanking the line which represents the edge, then

$$\sin^2 s = \cot \mathbf{A} \, . \cot \mathbf{B}$$

Once slopes are known, it is trivial to recover edge lengths and other parameters. Both Attneave and Frost and Perkins [4,5,6] produced evidence that the geometry of cubic corners is relevant to human vision.

A closely related assumption is that two angles at a vertex are right. This does not determine the slopes of the edges from a single view, but it does determine slopes when two views are available. Cowie [7] has described methods of recovering edge slope (and hence other parameters) in this case. The methods generalise to the situation where multiple views are available and it is not assumed that the non-right angle at the vertex is constant - this sitation arises, for instance, when a card is folded or unfolded, and it is related to the biological motion problem where

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two limbs pivot at right angles to an implicit link.

We have collected new psychophysical evidence which indicates that human vision pays particular attention to the case where two angles are right. Observers looked at computer generated vertices (each consisting of three edges meeting at a point). They were asked to judge the size of one of the three angles in each - call this the key angle. Observers were able to move the objects on the screen using a mouse. There were thirty objects in all. In ten cases, the two non key angles were right: in ten, the other two involved no particular regularity (one was 50° and the other was 70°). Absolute error was twice as large in the second condition, indicating that human vision can trade on the presence of paired right angles. Accuracy was not improved if the edges were equal in length, indicating that equality of length is not significant in the way that the presence of paired right angles is.

The remaining objects in the experiment indicate that human vision trades on another type of regularity, the presence of three equal angles at a vertex. In these objects both non-key angles were equal to the key angle. In this condition errors were significantly lower than they were when the non-key angles were irregular, and not significantly higher than errors is the case where the non-key angles were right. Again, it was irrelevant whether the arms were of equal lengths.

Another, quite different technique also indicates that human vision trades on the presence of equal angles at a vertex. Clements and Cowie [8] have studied the 'Rubber Rhomboid' effect. The effect occurs when observers view a skeleton parallelepiped rotating. Cowie [9] noted that with a non-rectangular parallelepiped, the structure appeared to deform as it moved, as if the visual system were imposing an assumption about the geometric form of the object at each instant - with the result that if the assumption did not fit then the observer's interpretation changed from instant to instant and the structure appeared non-rigid. Clements studied the amount of deformation that observers reported in a range of rotating parallelepipeds and plotted the results as a response surface (figure1). What it shows is that deformation was lowest when the three angles at the vertex were actually equal. The natural interpretation of that finding is that observers were imposing a presumption that angles would be equal.

These findings are consistent with a range of evidence that human vision has techniques for recovering 3-d structure which assume a considerable degree of geometric regularity in the environment without depending on fully specified models [10]. Evidence that techniques are used by human vision may not guarrantee that they will benefit machine vision, but it is a good reason to explore them. In the cases which we have outlined, the type of regularity which appears to be important involves the angles at a vertex. We therefore revisited the geometry of trihedral vertices with a view to understanding what could be derived from assumptions about angles between the edges.

MAPPING TRIHEDRAL VERTICES ONTO AN IMAGE

Our analysis begins by deriving a result which deals directly with the mapping of an object with known co-ordinates onto image points. Dhome et al. [1] proved essentially the same result by a different route (by deriving it from a more general relationship) and used it in a model matching routine. Our approach derives the key result directly, and in the processes establishes relationships which are useful in their own right. We then show that it can be used when the only information about the model involves the angles at the vertex, and we explore the way limiting cases constrain possible hypotheses about the angles at vertices.

Basic Equations

We start with a projection of a vertex as in figure 2. It contains three lines, and the angles between them are **A**, **B** . In the first instance we consider how to fit this to a vertex whose apex is at the origin and whose edges end at (1,0,0), (a,b,c) and (a',b',c'). This introduces information about edge lengths, but we will show later that this can be dispensed with.

To fit a vertex to the projection we begin with the edges of the vertex as illustrated in figure 3 in which $c,c' \geq 0$ and $b' \leq 0 \leq b$. We then rotate them through an angle $\mathbf{X}$ ($|\mathbf{X}| \leq 90°$) about the x-axis followed by a rotation about the y -axis through an angle $\mathbf{Y}$ ($|\mathbf{Y}| < 90°$) such that projection onto the xy -plane is as given in figure 2 in which $0° < \mathbf{A},\mathbf{B} < 180°$. Thus OP is transformed to OP_0 and OQ is transformed to OQ_0. These rotations are achieved by using the matrices:

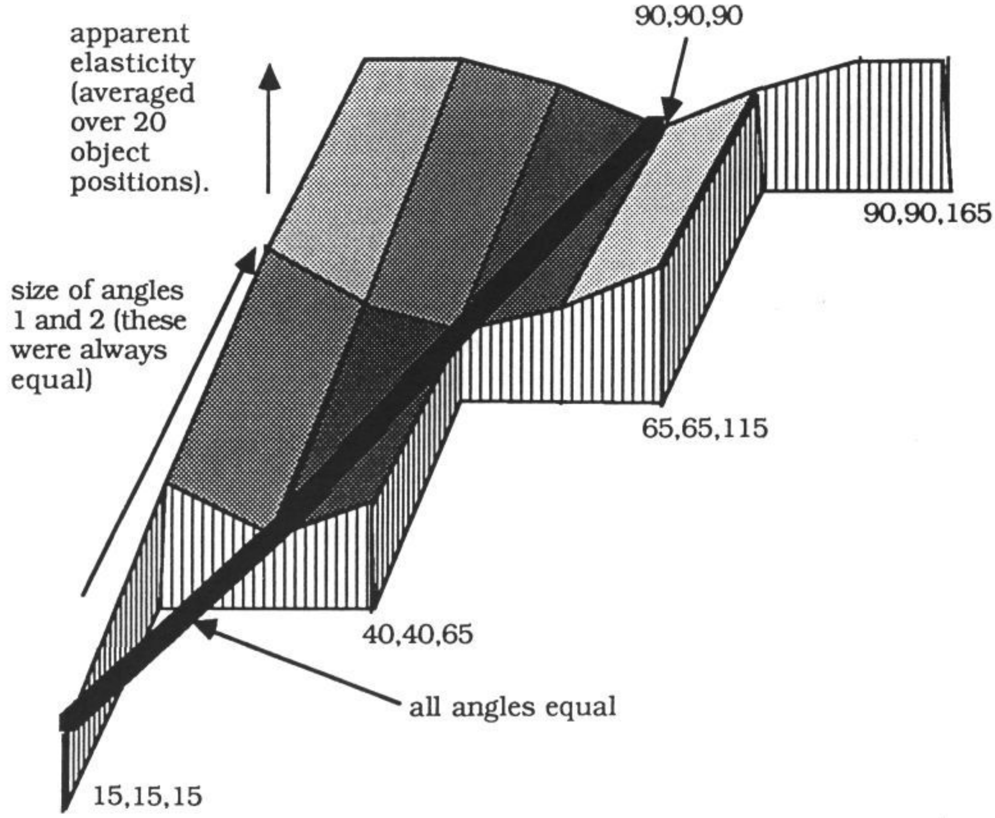

Figure 1 : apparent elasticity of rigidly rotating parallelepipeds as a function of the angles between edges: elasticity is least when the angles are equal.

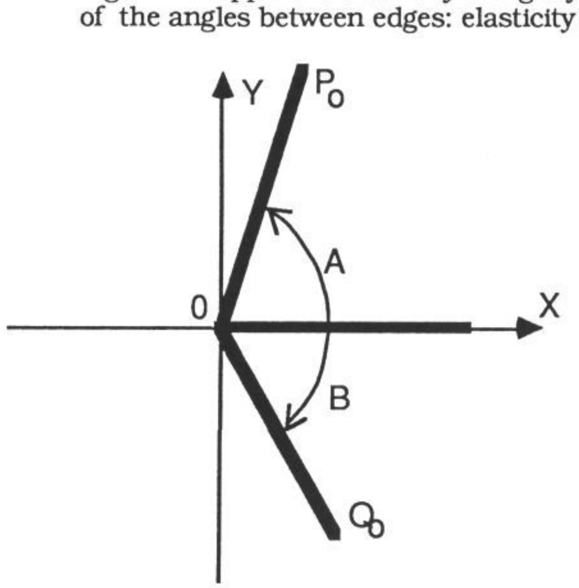

Figure 2: projection of a vertex

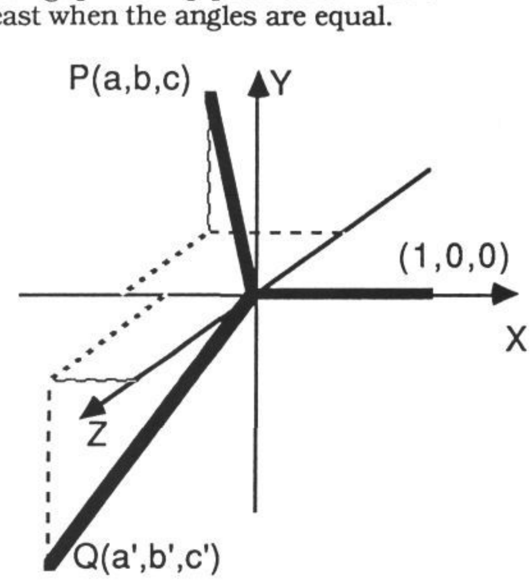

Figure 3: co-ordinates of a vertex

$$R_{\mathbf{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\mathbf{X} & \sin\mathbf{X} \\ 0 & -\sin\mathbf{X} & \cos\mathbf{X} \end{pmatrix}$$

and

$$R_{\mathbf{Y}} = \begin{pmatrix} \cos\mathbf{Y} & 0 & \sin\mathbf{Y} \\ 0 & 1 & 0 \\ -\sin\mathbf{Y} & 0 & \cos\mathbf{Y} \end{pmatrix}$$

and note that $R_{\mathbf{X}}$ is anticlockwise if $\mathbf{X} > 0$ and $R_{\mathbf{Y}}$ is clockwise if $\mathbf{Y} > 0$. We then obtain

$$(1,0,0)\, R_{\mathbf{X}} R_{\mathbf{Y}} = (\cos\mathbf{Y},\ 0,\ \sin\mathbf{Y})$$

$$(a,b,c)\, R_{\mathbf{X}} R_{\mathbf{Y}} = (a\cos\mathbf{Y} - \sin\mathbf{Y}(b\sin\mathbf{X} + c\cos\mathbf{X}),\ b\cos\mathbf{X} - c\sin\mathbf{X},\ a\sin\mathbf{Y} + \cos\mathbf{Y}(b\sin\mathbf{X} + c\cos\mathbf{X}))$$

$$(a',b',c')\, R_{\mathbf{X}} R_{\mathbf{Y}} = (a'\cos\mathbf{Y} - \sin\mathbf{Y}(b'\sin\mathbf{X} + c'\cos\mathbf{X}),\ b'\cos\mathbf{X} - c'\sin\mathbf{X},\ a'\sin\mathbf{Y} + \cos\mathbf{Y}(b'\sin\mathbf{X} + c'\cos\mathbf{X}))$$

the xy -coordinates of which are co -ordinates of P_0 and Q_0, and so

$$\{a\cos\mathbf{Y} - \sin\mathbf{Y}(b\sin\mathbf{X} + c\cos\mathbf{X})\}\sin\mathbf{A} = (b\cos\mathbf{X} - c\sin\mathbf{X})\cos\mathbf{A}$$
$$\{a'\cos\mathbf{Y} - \sin\mathbf{Y}(b'\sin\mathbf{X} + c'\cos\mathbf{X})\}\sin\mathbf{B} = -(b'\cos\mathbf{X} - c'\sin\mathbf{X})\cos\mathbf{B} \qquad (1)$$

Set
$a_s = a\sin\mathbf{A}$; $a'_s = a'\sin\mathbf{B}$; $b_s = b\sin\mathbf{A}$;
$b'_s = b'\sin\mathbf{B}$; $c_s = c\sin\mathbf{A}$; $c'_s = c'\sin\mathbf{B}$;
$b_c = b\cos\mathbf{A}$; $b'_c = b'\cos\mathbf{B}$; $c_c = c\cos\mathbf{A}$;
$c'_c = c\cos\mathbf{B}$.

Then eliminating $\cos\mathbf{Y}$ and $\sin\mathbf{Y}$ we obtain

$$(E\sin\mathbf{X} + F\cos\mathbf{X})\sin\mathbf{Y} = J\cos\mathbf{X} - K\sin\mathbf{X}$$
$$(E\sin\mathbf{X} + F\cos\mathbf{X})\cos\mathbf{Y} = L\sin\mathbf{X}\cos\mathbf{X} + M\cos^2\mathbf{X} - N\sin^2\mathbf{X}$$

where
$E = a_s b'_s - a'_s b_s$; $F = a_s c'_s - a'_s c_s$
$J = a_s b'_c + a'_s b_c$; $K = a_s c'_c + a'_s c_c$
$L = b_c b'_s + b'_c b_s - c_c c'_s - c'_s c_s$;
$M = b_c c'_s + b'_c c_s$; $N = b_s c'_c + b'_s c_c$

Now using $\sin^2\mathbf{Y} + \cos^2\mathbf{Y} = 1$ gives the quartic in $\tan\mathbf{X}$

$$(K^2 - E^2 + N^2)\tan^4\mathbf{X} - 2(EF + JK + LN)\tan^3\mathbf{X} + (K^2 - E^2 + J^2 - F^2 + L^2 - 2MN)\tan^2\mathbf{X} - 2(EF + JK - LM)\tan\mathbf{X} + (J^2 - F^2 + M^2) = 0 \qquad (2)$$

The roots of (2) give the values of **X** for which the linear equations (1) are consistent and no real roots means that the trihedral vertex cannot produce the given projection. Also if all the coefficients are zero either no **X** makes (1) consistent or infinitely many **X** make (1) consistent, but if $K^2 - E^2 + N^2 = 0$ and $J^2 - F^2 + M^2 \neq 0$ then $|\mathbf{X}| = 90$ (i.e. $\cot\mathbf{X} = 0$) is a solution. In these special cases **Y** can easily be found from equations (1) when it exists. For each real root we set $\cos\mathbf{X} = (1 + \tan^2\mathbf{X})^{-0.5}$ and $\sin\mathbf{X} = \cos\mathbf{X}\tan\mathbf{X}$ (as $|\mathbf{X}| < 90°$) which we then substitute into equations (1) to obtain the corresponding solutions for $\sin\mathbf{Y}$ and $\cos\mathbf{Y}$. If $E\sin\mathbf{X} + F\cos\mathbf{X} = 0$ then two sets of solutions for $\sin\mathbf{X}$ and $\cos\mathbf{Y}$ are obtained and these may not be real. Each corresponding set of real solutions for $\tan\mathbf{X}$, $\sin\mathbf{Y}$ and $\cos\mathbf{Y}$ in which $\cos\mathbf{Y} > 0$ can now be checked to see if they produce the required projection.

It is important to note that if we change the co-ordinates of (a, b, c) and (a', b', c') to $(\alpha a, \alpha b, \alpha c)$ and $(\beta a', \beta b', \beta c')$ where $\alpha, \beta > 0$ this produces the same equations (1). Thus the solutions for **X** and **Y** depend on the projection angles **A** and **B**, the angles between the edges in the initial position and the orientation of these edges. In fact only **X** depends on the orientation.

Thus initially we need not know the lengths of the initial edges. All we need are co-ordinates (a, b, c) and (a', b', c') which represent the directions of OP and OQ subject to a suitable orientation, and the lengths of the edges can be calculated from the lengths of the lines in the projection and the solutions for **X** and **Y**.

Two formulae for calculating normalised co - ordinates (a, b, c) and (a', b', c') are given as follows:

Let **U**, **V**, **W** be respectively the angles between *OP* and the x -axis, *OQ* and the x-axis, and *OP* and *OQ*. Set $a = \cos\mathbf{U}$, $a' = \cos\mathbf{V}$.

FORMULA I :

$$c^2 = c'^2 = \frac{1 - \cos^2\mathbf{U} - \cos^2\mathbf{V} - \cos^2\mathbf{W} + 2\cos\mathbf{U}\cos\mathbf{V}\cos\mathbf{W}}{2 - 2\cos\mathbf{W} - (\cos\mathbf{U} - \cos\mathbf{V})^2} \qquad (c, c' \geq 0)$$

If $bb' = \cos\mathbf{W} - \cos\mathbf{U}\cos\mathbf{V} - c^2 \leq 0$, set $b = \sqrt{\sin^2 \mathbf{U} - c^2}$ and $b' = -\sqrt{\sin^2 \mathbf{V} - c^2}$. Otherwise use formula II.

FORMULA II:

$$b^2 = b'^2 = \frac{1 - \cos^2\mathbf{U} - \cos^2\mathbf{V} - \cos^2\mathbf{W} + 2\cos\mathbf{U}\cos\mathbf{V}\cos\mathbf{W}}{2 + 2\cos\mathbf{W} - (\cos\mathbf{U} + \cos\mathbf{V})^2} \quad (b' \le 0 \le b)$$

If $cc' = \cos\mathbf{W} - \cos\mathbf{U}\cos\mathbf{V} + b^2 \ge 0$,

set $c = \sqrt{\sin^2\mathbf{U} - b^2}$ and $c' = \sqrt{\sin^2\mathbf{V} - b^2}$
Otherwise use formula I.

These relationships allow us to achieve one of our objectives, to input the angles at a specified vertex and to derive a match for a given image. We now consider the informative simplifications which arise in special cases.

Simplifications and limiting cases

If $\mathbf{U}=\mathbf{V}$, giving a symmetrical vertex, then formulae I and II are equivalent and so $\mathbf{X}$ is more closely related to the projection angles $\mathbf{A}$ and $\mathbf{B}$ in this case. In addition

$a = a' = \cos\mathbf{U}$, $b = -b' = \sin\mathbf{W}/2$,

$c = \sqrt{\sin^2\mathbf{U} - \sin^2(\mathbf{W}/2)}$

Consequently the quartic equation (2) becomes

$$\begin{aligned}
&\{a^2c^2\sin^2(\mathbf{A}+\mathbf{B}) + b^2c^2\sin^2(\mathbf{A}-\mathbf{B}) \\
&\quad - 4a^2b^2\sin^2\mathbf{A}\sin^2\mathbf{B}\}\tan^4\mathbf{X} \\
&+2bc\sin(\mathbf{A}-\mathbf{B})\sin(\mathbf{A}+\mathbf{B})\tan^3\mathrm{X} \\
&+\{a^2b^2\sin^2(\mathbf{A}-\mathbf{B}) + a^2c^2\sin^2(\mathbf{A}+\mathbf{B}) \\
&\quad - 4a^2b^2\sin^2\mathbf{A}\sin^2\mathbf{B} + (b^2+c^2)\sin^2(\mathbf{A}+\mathbf{B}) \\
&\quad + 2b^2c^2\sin^2(\mathbf{A}-\mathbf{B})\}\tan^2\mathrm{X} \\
&+2bc\sin(\mathbf{A}-\mathbf{B})\sin(\mathbf{A}+\mathbf{B})\tan\mathbf{X} \\
&+a^2b^2\sin^2(\mathbf{A}-\mathbf{B}) + b^2c^2\sin^2(\mathbf{A}-\mathbf{B}) = 0 \qquad (3)
\end{aligned}$$

In the special case where $\mathbf{U} = \mathbf{V} = 90^\circ$, we have $a=0$, $b^2 + c^2 = 1$ and so the quartic factorises to become

$$(bc\sin(\mathbf{A}-\mathbf{B})\tan^2\mathbf{X} + \sin(\mathbf{A}+\mathbf{B})\tan\mathbf{X} + bc\sin(\mathbf{A}-\mathbf{B}))^2 = 0$$

This gives a quadratic whose discriminant is

$$\begin{aligned}
&\sin^2(\mathbf{A}+\mathbf{B}) - 4b^2c^2\sin^2(\mathbf{A}-\mathbf{B}) \\
&= \sin^2(\mathbf{A}+\mathbf{B}) - \sin^2\mathbf{W}\sin^2(\mathbf{A}-\mathbf{B}) \\
&\qquad (\text{as } b = \sin\mathbf{W}/2)
\end{aligned}$$

This defines a limiting condition

$$|\mathbf{A}-\mathbf{B}| \le \sin^{-1}(\sin(\mathbf{A}+\mathbf{B})/\sin\mathbf{W})$$

Like Perkins' rule for cubic corners, this allows us to set bounds on possible hypotheses about the angles in a vertex with minimal computation.

Continuing with the case where $\mathbf{U}=\mathbf{V}=90^\circ$, let us now return to the initial position of the trihedral vertices and rotate through an angle $\mathbf{X}$ ($|\mathbf{X}| \le \mathbf{W}/2$) about the x-axis. By rotating now about the y-axis we see that $\mathbf{W} \le \mathbf{A}+\mathbf{B} \le 360^\circ - \mathbf{W}$. When $\mathbf{W} \le 90^\circ$, $\mathbf{A}$ and $\mathbf{B}$ are acute if $\mathbf{A}+\mathbf{B} < 180^\circ$ and they are obtuse if $\mathbf{A}+\mathbf{B} > 180^\circ$. This is directly analogous to Perkins' rule.

Let us now examine the general symmetric case where $\mathbf{A}+\mathbf{B} = 180^\circ$. In this case (3) reduces to

$$\begin{aligned}
&(c^2\cos^2\mathbf{A} - a^2\sin^2\mathbf{A})\tan^4\mathbf{X} \\
&+(a^2\cos^2\mathbf{A} - a^2\sin^2\mathbf{A} + 2c^2\cos^2\mathbf{A})\tan^2\mathbf{X} \\
&+(a^2+c^2)\cos^2\mathrm{A} \\
&= (1+\tan^2\mathbf{X})((c^2\cos^2\mathbf{A} - a^2\sin^2\mathbf{A})\tan^2\mathbf{X} \\
&\qquad + (a^2+c^2)\cos^2\mathbf{A}) \\
&= 0
\end{aligned}$$

Thus

$$\tan^2\mathbf{X} = (a^2+c^2)\cos^2\mathbf{A}/(a^2\sin^2\mathbf{A} - c^2\cos^2\mathbf{A})$$

But the edges cannot be rotated about the x-axis so that P and Q are to one side of the xz-plane and so $|\tan\mathbf{X}| \le b/c$. Thus

$$c^2(a^2+c^2)\cos^2\mathbf{A} \le b^2(a^2\sin^2\mathbf{A} - c^2\cos^2\mathbf{A})$$

i.e.

$$a^2b^2\sin^2\mathbf{A} \ge c^2(a^2+b^2+c^2)\cos^2\mathbf{A} = c^2\cos^2\mathbf{A}$$

so

$$\tan^2\mathbf{A} \ge c^2/a^2b^2 = (\sin^2\mathbf{U} - \sin^2\mathbf{W}/2)/\cos^2\mathbf{U}\sin^2\mathbf{W}/2$$

This formula establishes limiting cases for each of $\mathbf{A}$, $\mathbf{U}$ or $\mathbf{W}$ depending on the problem we are wanting to examine. Other limiting cases can easily be derived from equations (1) in both the symmetric and non-symmetric cases by rotating the trihedral edges through $\mathbf{X} = \tan^{-1}(b'/c')$ or $\mathbf{X} = \tan^{-1}(b/c)$ about the x-axis and then examining special cases for $\mathbf{Y}$. This amounts to placing P or Q in the xz-plane.

CONCLUSION

As they stand, these results allow us to tackle specific problems. For instance, we are currently using them as a basis for a program which takes projections of rotating objects like Clements' 'Rubber Rhomboids' and generates interpretations which contain three equal angles. The program generates a family of interpretations, one for each possible value of the angle at the apex, from each image. One can then select among these alternatives by requiring relative consistency about lengths from one instant to the next. This is closely akin to the problems tackled in Cowie's treatment of right angles [7], and similar techniques are appropriate. An important extension is to use limiting conditions to identify the range of options which is consistent with the values of **A** and **B** that have occurred during rotation.

More generally, the results are a basis upon which to develop a full analysis of trihedral corners. We can examine the simplifications which occur not just in the few special cases which have been studied above, but in a wide range. These include equalities of length as well as angle.

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