Radon transformation of δ-function Curves.
A Geometric Approach.

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Abstract

The Radon transformations of generalized functions concentrated on lines and curves are deduced using a geometric approach and a limiting process. It is shown that any curve may be replaced by its tangents for the purposes of transformation and that each tangent will produce a maximum value in transform space which may be located using a one dimensional convolution filter. The locations of such maximum values may then be used to deduce the equation of the curve in image space.

1 . Introduction

The present work addresses the problem of evaluating the Radon transforms [1] of generalized functions concentrated on lines and curves. The application of the work is in the field of computer vision where it has been shown [2] that shape primitives in an edge image may be uniquely characterized by their shape indicative distributions in a two dimensional Radon transform space. For simplicity attention is focused here on the case of a binary edge image where the shape primitives in the edge image may be represented mathematically by unit density delta functions, the results however are easily generalized.

A great deal of theoretical work has been undertaken by the mathematicians, most notably Gel'fand, Graev and Vilenkin [3]; much of this work is directly applicable to the task of shape detection in computer vision but remains inaccessible to scientists without advanced mathematical training. An attempt is made to bridge the gap between theory and application by interpreting the available analytical work in terms of geometric propositions which are perhaps easier to follow and which give the potential user a more practical understanding of the Radon transform. Many aspects of the technique not immediately obvious in a purely analytical treatment become clear when expressed geometrically, and the need to evaluate complicated integrals is avoided.

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2 . The Radon Transform.

The Radon transform may be written in the convenient form suggested by Gel'fand et al. [3]

$$\Re\{F(\mathbf{x})\} = H(p,\xi) = \int_{-\infty}^{\infty} d\mathbf{x}\, F(\mathbf{x})\delta(p-\xi\cdot\mathbf{x}) \quad (1)$$

In two dimensions the delta function, $\delta(p-\xi\cdot\mathbf{x})$, represents a line, L, of infinite length. ξ is a unit vector in the direction of the normal to that line and p is the algebraic length of the normal. It is of particular interest to consider the case in which the general function $F(\mathbf{x})$ is replaced by a particular function $F_D(\mathbf{x})$, where

$$F_D(\mathbf{x}) = \begin{cases} 1, & \text{in } D; \\ 0, & \text{otherwise.} \end{cases}$$

Fig 1 illustrates the Radon transform of such a function. The shaded region represents the function $F_D(\mathbf{x})$. The line L acts as a probe or detector function and $F_D(\mathbf{x})$ as the object function. Whenever the line L and the domain D intersect the value of the integral is equal to the length of the intersection; otherwise it is zero.

The above definition leads quite naturally to the introduction of a shape descriptive element. The domain D may be taken to be a narrow band

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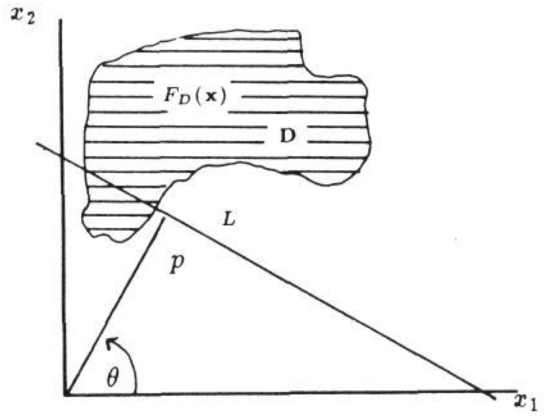

Fig. 1 Graphical representation of the Radon Transform.

of uniform width whose shape is that of the curve of interest. A limiting process may then be applied to obtain a distribution concentrated along the curve of interest.

3 . Generalized function concentrated on a straight line

Consider the case in which the domain D of the function $F_D(\mathbf{x})$ is an infinitely long vertical strip, $S(h)$, of width h, centred on the line $x_1 = x_{1_0}$; see Fig 2 .

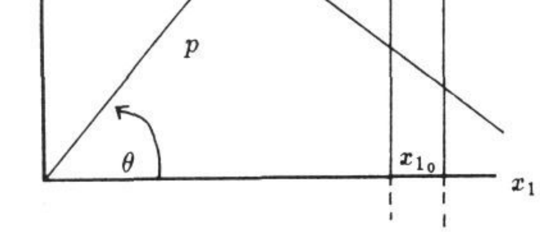

Fig. 2 Radon Transform of a thin vertical strip.

Then

$$F_{S(h)}(\mathbf{x}) = \begin{cases} 1, & |x_1 - x_{1_0}| \leq \frac{1}{2}h; \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

and its Radon transform (here simply the length of the intersection of L with the strip) is given by

$$\Re\{F_{S(h)}(\mathbf{x})\} = \frac{h}{|\sin\theta|}. \tag{3}$$

Now let us introduce a weighting factor of $1/h$ and consider the function

$$F_h(\mathbf{x}) = F_{S(h)}(\mathbf{x})/h, \tag{4}$$

which has everywhere unit density per unit length along the strip, whatever the value of h. Then from the above result we have

$$\Re\{F_h(\mathbf{x})\} = \frac{1}{|\sin\theta|}. \tag{5}$$

Since this expression is independent of h it remains valid in the limit as $h \to 0$, the case of the unit-density δ-function distribution along the line $x_1 = x_{1_0}$. Thus we have shown that

$$\Re\{\delta(x_1 - x_{1_0})\} = \frac{1}{|\sin\theta|}. \tag{6}$$

Deans [4] arrives at the same result, basing his analysis on the work of Gel'fand et al. [3]. It is simple to show [6] using the properties of shifting and linear transformation, that the corresponding result for a line whose normal subtends an angle ψ with the x_1 axis is

$$\Re\{\delta(p_1 - x_1\cos\psi - x_2\sin\psi)\} = \frac{1}{|\sin(\theta - \psi)|} \tag{7}$$

We note that the standardization of the equation of an arbitrary line to the form

$$p_1 = x_1\cos\psi + x_2\sin\psi \tag{8}$$

ensures that the corresponding δ-function distribution has unit linear density and is necessary for the validity of the result.

4 . The general case

The result obtained in section 3 can be used to deduce the Radon transform of any unit density δ-function distribution lying along any smooth curve C. It is convenient to denote this function by $\delta(C)$. The only contribution to the integral defining the transform comes from the points of intersection of C with the scanning line L, and for the purposes of computation the curve can be replaced by its tangent in the neighbourhood of such points. Let us suppose that for given values of p and θ there are n points of intersection, at which the normals to C make angles $\psi_j\,(j = 1, .., n)$ with the x_1 axis; see Fig 3 . Then according to equation 7 we have

$$\Re\{\delta(C)\} = \sum_{j=1}^{n} \frac{1}{|\sin(\theta - \psi_j)|} \qquad (9)$$

From this result it is clear that the essence of the problem is the determination of the points of intersection and the associated angles ψ_j.

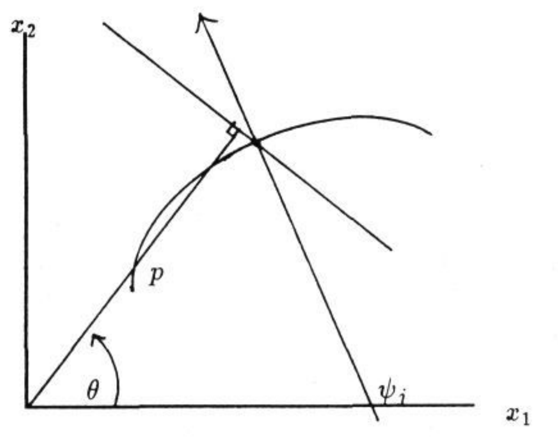

Fig. 3 Contribution to the Radon Transform of an arbitrary point on a curve.

Now let us suppose that the curve C is defined by an equation $f(\mathbf{x}) = 0$. In order to fulfil the unit-density condition we need to consider the transform not of $\delta\{f(\mathbf{x})\}$ but of $\delta\{w(\mathbf{x})f(\mathbf{x})\}$ where

$$w(\mathbf{x}) = \frac{1}{|\mathrm{grad} f(\mathbf{x})|} \qquad (10)$$

is a weighting function defined on C; see [6]. Explicit determination of $w(\mathbf{x})$ is unnecessary in practice, however, since the unit-density property is automatically realized by expressing the equations for the tangents in the normalized form of equation 7. For completeness we rewrite the result, equation 11, in terms of $f(\mathbf{x})$ giving

$$\Re\{\delta(w(\mathbf{x})f(\mathbf{x}))\} = \sum_{j=1}^{n} \frac{1}{|\sin(\theta - \psi_j)|} \qquad (11)$$

From the fundamental result, (9) (equivalently (11)), it is evident that the Radon transform of any tangent to the curve is singular at $\theta = \psi_j$. Such singular points may be detected using a one dimensional convolution filter [6]. To illustrate, the binary image of a parabola is shown in Fig. 4. An intensity map of the transform plane of Fig. 4 is shown in Fig. 5; whilst the singular points detected in Fig. 5 after the application of a one dimensional filter are shown in Fig. 6. From our knowledge of the location of these points we are able to reconstruct the curve C as the envelope of the tangents. Fig. 7 illustrates this process. Each value of p and θ located in the transform plane by the convolution filter can be used to deduce the equation of a tangent to the curve in image space. In the reconstruction the points of intersection of the tangents will be most dense in the neighbourhood of the curve (shown in black) and hence maximum intensities are observed in this region.

5 . Application to an ellipse

As an illustration of the method described above we consider the case of an ellipse, E, defined by the equation

$$\frac{(x_1)^2}{a^2} + \frac{(x_2)^2}{b^2} = 1 \qquad (12),$$

The probe line of equation 1 may be written as:

$$p = \xi \cdot \mathbf{x} = x_1 \cos\theta + x_2 \sin\theta \qquad (13)$$

An arbitrary point, P_ϕ, on E may be expressed as:

$$(x_1\, ,\, x_2) = (a\cos\phi\, ,\, b\sin\phi) \qquad (14)$$

where ϕ is the parametric angle. In order to determine the values of ϕ corresponding to points at which L intersects E we substitute from equation 14 into equation 13 to obtain

$$p = a\cos\theta\cos\phi + b\sin\theta\sin\phi \qquad (15)$$

This expression may be rewritten in the form

$$p = M \cos(\phi - \chi) \tag{16}$$

where

$$\left.\begin{aligned} \cos\chi = \frac{a\cos\theta}{M} \quad , \quad \sin\chi = \frac{b\sin\theta}{M} \\ M = (a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}} \end{aligned}\right\} \tag{17}$$

Hence the required values of ϕ are

$$\phi_1, \phi_2 = \chi \pm \cos^{-1}\left(\frac{p}{M}\right) \tag{18}$$

provided that $|p| \leq M$. (If $|p| > M$ no intersection occurs.)

Now the equation of the tangent to the ellipse at P_ϕ is given by

$$\frac{x_1}{a}\cos\phi + \frac{x_2}{b}\sin\phi = 1 \tag{19}$$

which on reduction to the normalized form of equation 7 becomes

$$p_1 = x_1\cos\psi + x_2\sin\psi \tag{20}$$

where

$$\left.\begin{aligned} p_1 = \frac{1}{K}, \quad \cos\psi = \frac{\cos\phi}{aK}, \quad \sin\psi = \frac{\sin\phi}{aK} \\ K = \left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)^{\frac{1}{2}} \end{aligned}\right\} \tag{21}$$

With ψ_1, ψ_2 defined by setting $\phi = \phi_1, \phi_2$ respectively in equation 21 we now obtain, using equation 11, the desired Radon transform

$$\Re\{\delta(E)\} = \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{1}{|\sin(\theta - \psi_j)|}, & \text{when } |p| \leq M; \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

With the aid of equations 17,18 and 21 it is readily shown that

$$\frac{1}{|\sin(\theta - \psi_j)|} = \frac{abK}{(M^2 - p^2)^{\frac{1}{2}}} \tag{23}$$

thus demonstrating the equivalence of the expression, equation 22, with results obtained elsewhere; see [6].

6 . Conclusions.

It has been shown that the Radon transforms of δ-function curves may be deduced using a geometric approach and a limiting process. The resulting expressions are in agreement with those derived using purely analytical methods [6]. As a consequence of this treatment it becomes obvious that any curve may be replaced by its tangents for the purposes of transformation and each tangent will produce a maximum value in transform space which may be located using a one dimensional convolution filter [6].

For simplicity, only binary edge images have been considered and hence attention is focused on the case of unit density delta functions. The results may however be easily extended to deal with the more general case where edge strengths may need to be considered.

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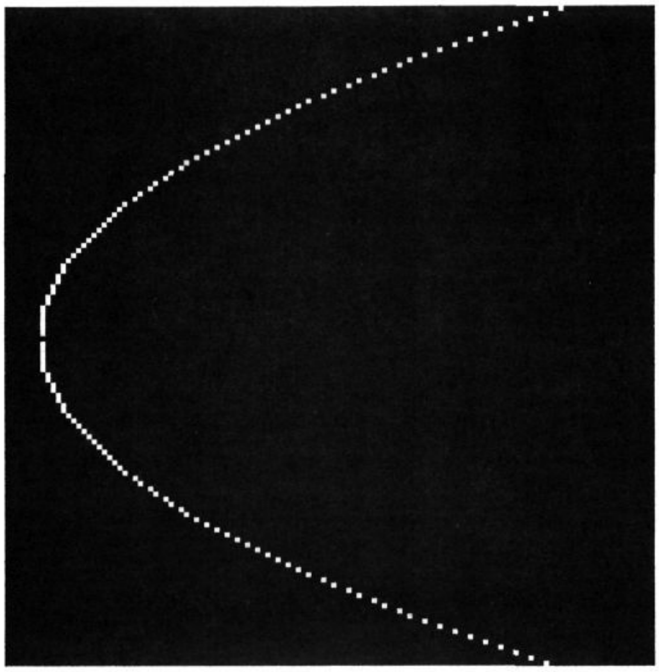

Fig. 4 Binary Image of a Parabola.

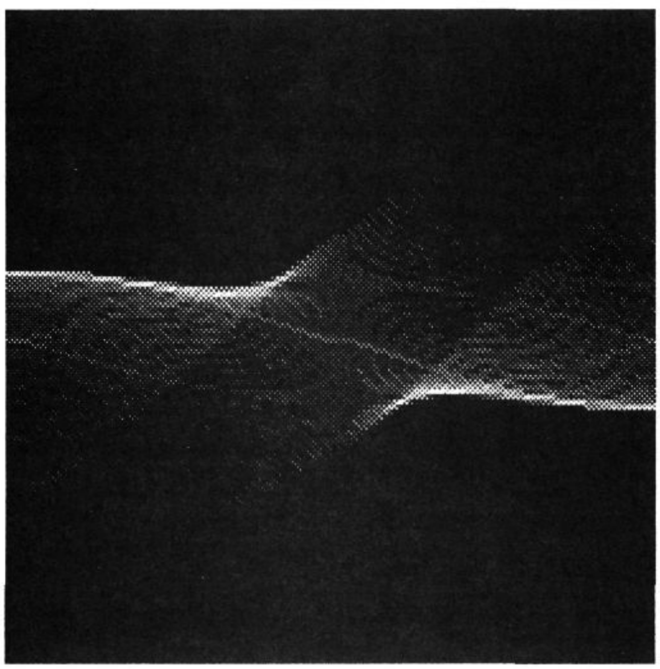

Fig. 5 Transform of Parabola.

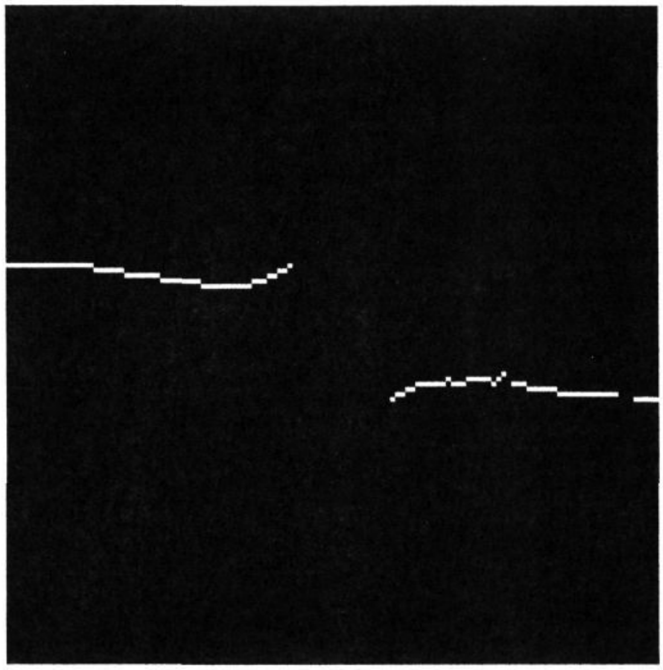

Fig. 6 Singular Points Detected.

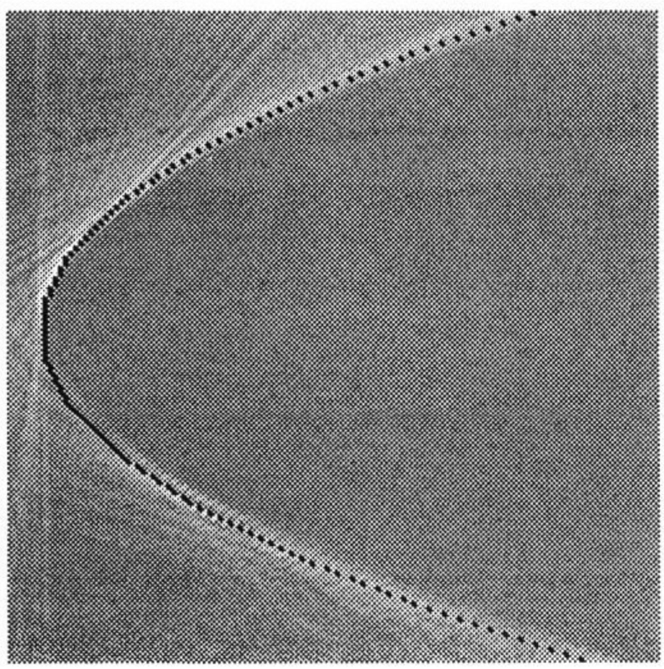

Fig. 7 Reconstruction of the curve